

Langevin dynamics with a tilted periodic potential: a model for randomly sized avalanches

Gioia Carinci^(a) and Stephan Luckhaus^(b)

^(a) *University of Modena and Reggio Emilia, via G. Campi 213/b, 41125 Modena, Italy*
e-mail: gioia.carinci@unimore.it

^(b) *University of Leipzig, Augustus Platz, 10-11, D-04109 Leipzig, Germany*
e-mail: luckhaus@mis.mpg.de

August 9, 2012

Abstract

We study a Langevin equation for a particle moving in a periodic potential in the presence of viscosity γ and subject to a further external field α . For a suitable choice of the parameters α and γ the related deterministic dynamics yields heteroclinic orbits. In such a regime, in absence of stochastic noise both confined and unbounded orbits coexist. We prove that, with the inclusion of an arbitrarily small noise only the confined orbits survive in a sub-exponential time scale.

1 Motivations

In many physical contexts e.g. the switching of magnetic domain walls (Barkhausen noise) or the motion of twin boundaries in crystals one observes an intermittent dynamics of energy relaxation with “relaxation events” of random amplitude. This type of dynamics is commonly called Avalanche Dynamics (see e.g. [6]).

Our motivation is to give a possible explanation for this kind of mechanism. We need three ingredients: a rough interaction potential with many local energy barriers, and a small tilt; an “almost Hamiltonian” dynamics that approximately conserves the total energy, helping the system to overcome the next energy barrier once it passes the first one; finally a weak coupling to a heat bath, modelled in our case by a small Langevin noise.

As a proof of principle we investigate in this paper the simplest such system in a two dimensional phase space. We show in a suitable singular limit that the number of consecutive crossed energy barriers is indeed random. The intermittency structure, i.e. the large deviation theory for the model system is not dealt with in this article and will be the subject of a future work.

2 Introduction

We consider a one-dimensional Brownian particle moving in a periodic oscillating potential $V_0(x)$ (e.g. $V_0(x) = \cos x$) in the presence of viscosity γ and subject to an additional constant external force α . The dynamics for the coordinate $x(t) \in \mathbb{R}$ of the particle is governed by the Langevin equation:

$$\ddot{x} + \gamma \dot{x} + V_0'(x) = \alpha + \epsilon \dot{w}(t) \quad (2.1)$$

where $\dot{w}(t)$ is the white noise, and ϵ a small parameter.

We denote by $V(x)$ the total potential taking into account also the linear term due to the external force, $V(x) = V_0(x) - \alpha x$. The equation of the motion (2.1) then becomes

$$\ddot{x} + \gamma \dot{x} + V'(x) = \epsilon \dot{w}(t) \quad (2.2)$$

and the related first order system is

$$\begin{cases} \dot{x} = p \\ \dot{p} = -\gamma p - V'(x) + \epsilon \dot{w}. \end{cases} \quad (2.3)$$

According to the values of the parameters and the initial conditions, the particle may escape in the direction of the force α or be trapped for a long time in one of the wells of the potential. Without any noise ($\epsilon = 0$), when α is large enough there are only “running solutions”, i.e. unbounded solutions. When the force α is small enough and the friction parameter γ is large enough, the particles finally reach one of the minima of the potential. In this case there are only “locked solutions”. For α and γ small enough, both types of solution coexist. With the addition of the noise there are certainly transitions between the locked and the running states in the exponential time scale. We wonder whether in certain critical regimes (in particular the critical regime where deterministic heteroclinic orbits exist) such transitions enabled by small stochastic fluctuations take place in a faster time scale.

Deterministic orbits

Consider the deterministic system related to (2.3), i.e. the $\epsilon = 0$ case,

$$\begin{cases} \dot{X} = P \\ \dot{P} = -\gamma P - V'(X) \end{cases} \quad (2.4)$$

and sketch the phase diagram. In each period of the original potential $V_0(x)$ there are confined and escaping orbits. The picture changes according to α and γ .

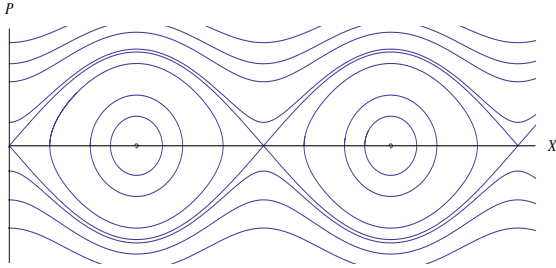


Figure 1: Phase diagram for $\alpha = \gamma = 0$. The orbits are periodic.

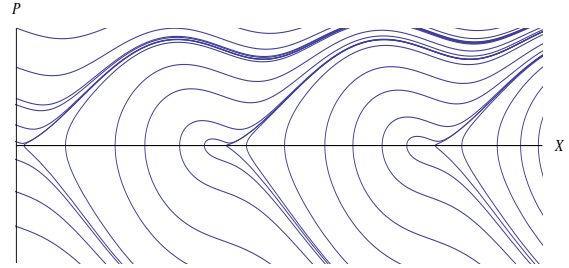


Figure 2: Phase diagram for $\alpha > 1$. There are only running orbits.

In absence of viscosity and external force (i.e. for $\gamma = \alpha = 0$), the orbits are periodic (see Figure 1). For positive γ and α the dynamics loses its periodicity. The inclusion of the friction makes the system dissipative, the particles lose energy, and the confined orbits are attracted by the local minima of the potential, whereas the running solutions have an asymptotic effective velocity.

Figures 2, 3 and 4 provide three possible phase diagrams. For $\alpha > 1$ the total potential $V(x)$ does not have local minima and then there are only running solutions (see Figure 2).

When $\alpha < 1$, there are also bounded solutions. For any $\gamma > 0$, there exists a critical value $\alpha_\gamma > 0$ such that, for $0 < \alpha < \alpha_\gamma$, there are only confined orbits (see Figure 3). For $\alpha_\gamma < \alpha < 1$ running and bounded orbits coexist (see Figure 4).

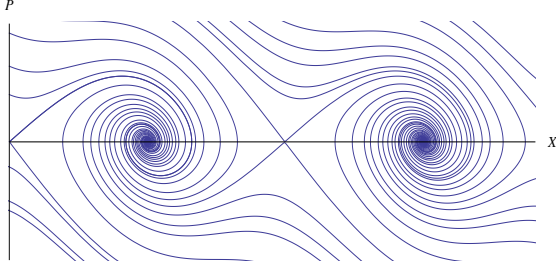


Figure 3: Phase diagram for $\gamma > 0$ and $0 < \alpha < \alpha_\gamma$. There are only confined orbits.

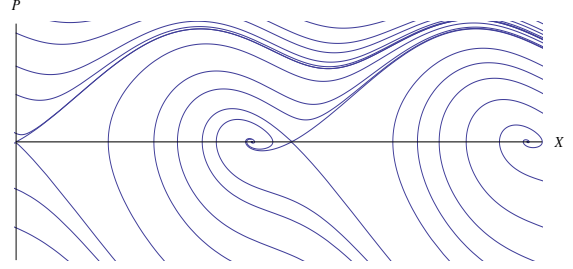


Figure 4: Phase diagram for $\gamma > 0$ and $\alpha_\gamma < \alpha < 1$. There are both running and confined orbits.

The situation is well resumed in Figure 5 that shows the dependence on γ of the critical value α_γ . As we will see below, α_γ is of the order of γ as $\gamma \rightarrow 0$.

For any $\alpha < 1$ there exist critical orbits, i.e. orbits asymptotically converging to some saddle equilibria (corresponding to some local maxima of the potential) in the phase plane. The critical scaling $\alpha = \alpha_\gamma$ is the one that gives rise to heteroclinic orbits, i.e. orbits connecting two consecutive local maxima of the potential.

Each orbit can be piecewise expressed by a function in the phase plane that we will usually denote by $\varphi(x)$. We say that $\varphi(x)$ is an orbit of our dynamics if there exists a solution $(X(t), P(t))$ of (2.4) and a suitable time interval $I \subseteq (0, +\infty)$ such that $\varphi(X(t)) = P(t)$ for any $t \in I$. $\varphi(x)$ must verify the

$$\frac{d\varphi}{dx}(x) = -\gamma - \frac{V'(x)}{\varphi(x)} \quad (2.5)$$

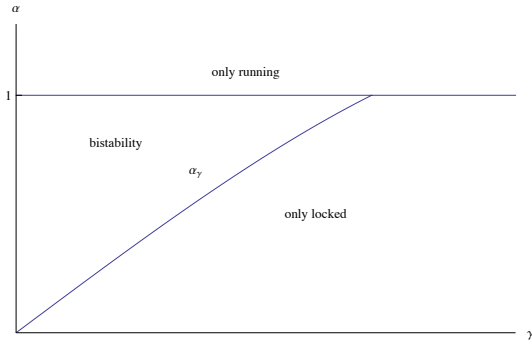


Figure 5: The picture shows the three different regimes in the plane (γ, α) .

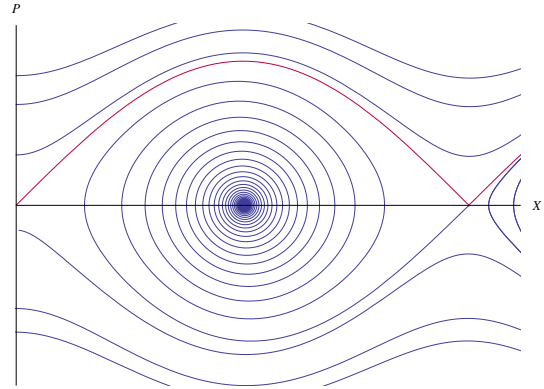


Figure 6: When $\alpha = \alpha_\gamma$ the heteroclinic orbit $\varphi_k^*(x)$ (red line) separates the escaping and the confined solutions.

The problem

In the present paper we are interested in the critical regime ($\alpha = \alpha_\gamma$) dynamics in the small noise limit ($\epsilon \rightarrow 0$). Thus from now on we fix γ small enough and $\alpha = \alpha_\gamma$. We denote by $\varphi_k^*(x)$ the k -th heteroclinic orbit, i.e. the orbit connecting the $k-1$ -th maximum of the potential with the k -th one:

$$\lim_{x \rightarrow 2(k-1)\pi^+} \varphi_k^*(x) = \lim_{x \rightarrow 2k\pi^-} \varphi_k^*(x) = 0 \quad (2.6)$$

see Figure 6.

In this paper we are not concerned with large deviations. We investigate the problem in a sub-exponential time scale. Far from the critical orbits we expect that the noise does not macroscopically affect the deterministic dynamics in such a fast time scale. On the other hand, there may be a macroscopic perturbation of the deterministic dynamics in a neighbourhood of the heteroclinic orbits. Then we choose the initial value to lay on one of the heteroclinic orbits, i.e. we denote by $(x(t), p(t))$ the solution of the problem

$$\begin{cases} \dot{x} = p & x(0) = -\pi \\ \dot{p} = -\gamma p - V'(x) + \epsilon \dot{w} & p(0) = \wp_0^*(-\pi), \end{cases} \quad (2.7)$$

and study the probability law of $(x(t), p(t))$ in the limit as $\epsilon \rightarrow 0$.

We show that, at each time, the probability for the particle, to get across the next well is $1/2$, in the limit as $\epsilon \rightarrow 0$. We prove, thus, that for fixed γ small enough, the random variable associated to the number of wells passed by $(x(t), p(t))$ has, for vanishing ϵ , a geometric distribution of parameter $1/2$. This implies, in particular, that the particle will finally be trapped in one of the wells for a long time, with probability 1 as $\epsilon \rightarrow 0$. The involved time scale, or more precisely, the velocity the particle travels before to be trapped in one of the wells is of order $\ln \epsilon$. With the inclusion of the noise, thus, the bistability between the locked and the running states is lost and only the locked state survives in this “fast time scale”.

Notation

Before turning to the precise statement, we introduce some notations. We shall use

- $f(\delta) = o(g(\delta))$ as $\delta \rightarrow 0$ if $\lim_{\delta \rightarrow 0} f(\delta)/g(\delta) = 0$;
- $f(\delta) = \mathcal{O}(g(\delta))$ as $\delta \rightarrow 0$ if there exists $c > 0$ such that $|f(\delta)| \leq c|g(\delta)|$ for any sufficiently small δ ;
- $f(\delta) = \Theta(g(\delta))$ as $\delta \rightarrow 0$ if there exist $c_1, c_2 > 0$ such that $c_1|g(\delta)| \leq |f(\delta)| \leq c_2|g(\delta)|$ for any sufficiently small δ ;
- $X \sim \mu$ if the stochastic variable has probability law μ .

Basic parameters

Although our results do not depend on the exact form of the potential, for the sake of simplicity we choose

$$V(x) = \cos(x - x_\alpha) - \alpha(x - x_\alpha), \quad x_\alpha = \arcsin \alpha$$

in such a way that $V(x)$ attains its local maxima at $x = 2k\pi$ and its local minima at $x = (2k-1)\pi + 2x_\alpha$. Let us investigate how α_γ is related to γ . All the orbits $\wp_k^*(x)$ must verify the equation (2.5), then

$$\frac{1}{2} \frac{d}{dx} (\wp_k^*(x))^2 = -V'(x) - \gamma \wp_k^*(x). \quad (2.8)$$

By integrating (2.8) in $(2(k-1)\pi, 2k\pi)$, and using the asymptotic conditions (2.6) we get

$$2\pi \frac{\alpha_\gamma}{\gamma} = \int_{2(k-1)\pi}^{2k\pi} \wp_k^*(x) dx = \Theta(1) \quad \text{as} \quad \gamma \rightarrow 0$$

thus $\alpha_\gamma = \Theta(\gamma)$ in the limit as $\gamma \rightarrow 0$.

We compute the asymptotic values of $d\wp_k^*(x)/dx$ for $x \rightarrow 2k\pi^\pm$. Let

$$\beta = \beta_\gamma := -V''(2k\pi) = \sqrt{1 - \alpha_\gamma^2} > 0 \quad k \in \mathbb{Z}, \quad (2.9)$$

then, from (2.5) we obtain that the asymptotic slopes of the heteroclinic orbits:

$$\lambda_+ := \frac{d}{dx} \wp_k^*(2(k-1)\pi^+) \quad \text{and} \quad \lambda_- := \frac{d}{dx} \wp_k^*(2k\pi^-) \quad (2.10)$$

must satisfy the equation

$$\lambda^2 + \gamma\lambda - \beta = 0,$$

then

$$\lambda_\pm = \lambda_\pm^\gamma = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\beta_\gamma}}{2} \quad (2.11)$$

notice that $\lim_{\gamma \rightarrow 0} \beta_\gamma = 1$ and $\lim_{\gamma \rightarrow 0} \lambda_\gamma^\pm = \pm 1$. We define, moreover, the parameter

$$\theta = \theta_\gamma := \frac{|\lambda_-|}{\lambda_+} - 1 = \frac{2\gamma}{\sqrt{\gamma^2 + 4\beta} - \gamma} \quad (2.12)$$

then $\theta_\gamma = \Theta(\gamma)$ as $\gamma \rightarrow 0$ with $\lim_{\gamma \rightarrow 0} \theta_\gamma/\gamma = 1$.

Critical region dynamics

In a neighborhood of the criticalities, i.e. when $x(t)$ approaches $2k\pi$, $V'(x(t))$ is well approximated by $-\beta(x(t) - 2k\pi)$, since $V'(2k\pi) = 0$. Hence the dynamics can be approximated by the linear system

$$\begin{cases} \dot{x} = p \\ \dot{p} = -\gamma p + \beta(x - 2k\pi) + \epsilon \dot{w} \end{cases} \quad (2.13)$$

A convenient choice of variables is given by

$$\begin{cases} z_k(t) := p(t) - \lambda_-(x(t) - 2k\pi) \\ v_k(t) := p(t) - \lambda_+(x(t) - 2k\pi) \end{cases} \quad (2.14)$$

with λ_\pm as defined in (2.11). Since in these variables the linearized system (2.13) becomes

$$\begin{cases} \dot{z} = \lambda_+ z + \epsilon \dot{w} \\ \dot{v} = \lambda_- v + \epsilon \dot{w} \end{cases} \quad (2.15)$$

where the equations are coupled only per the stochastic term.

Since $\wp_k^*(2k\pi) = 0$, by (2.10), for $|x(t) - 2k\pi|$ small enough we have

$$z_k(t) = p(t) - \wp_k^*(x(t)) + \mathcal{O}((x(t) - 2k\pi)^2) \quad \text{for} \quad x(t) < 2k\pi \quad (2.16)$$

and

$$v_k(t) = p(t) - \wp_{k+1}^*(x(t)) + \mathcal{O}((x(t) - 2k\pi)^2) \quad \text{for} \quad x(t) > 2k\pi. \quad (2.17)$$

The variable $|z_k(t)|$ can, thus, be thought of as a measure of the distance, in the phase plane, from the k -th heteroclinic orbit $\wp_k^*(\cdot)$ just before the k -th criticality. Equivalently $|v_k(t)|$ is a measure of the distance from the $k+1$ -th heteroclinic orbit $\wp_{k+1}^*(\cdot)$ just after the k -th criticality.

Stopping times

Far from criticalities the dynamics is stable and we expect the distribution of our process to be quite concentrated in a neighborhood of some deterministic paths (see the deterministic system (2.4)).

One of the main technicalities we run into in the proof of our result is the choice of the length of the critical interval. It is clear that the linear system (2.13) is a good approximation of our dynamics as long as $x(t)$ is close enough to the k -th criticality $2k\pi$. This provides an upper bound on the critical interval length. On the other hand, we need a good localization of the process at the beginning of the critical interval. We want the process to be still quite concentrated and not to diffuse too much. This clearly requires a lower bound on the length of the critical interval.

We denote by η_ϵ the order of magnitude of the length of the critical interval. Then, in order to make valid our approximations we need to impose the following condition on η_ϵ :

$$\eta_\epsilon = \epsilon^\nu \quad \text{with} \quad \frac{(1+\theta)^2}{3+2\theta} < \nu < \frac{1}{2} \quad (2.18)$$

and θ as in (2.12).

We define two sequences of stopping times. We call S_k the first time the process gets into the k -th critical interval and by T_{k+1} the first exit time from the k -th critical interval. The rigorous definition is given as follows

$$\begin{aligned} S_k &:= \inf\{t \geq 0 : v_k(t) \leq \eta_\epsilon\}, \quad k \geq 0 \\ T_k &:= \inf\{t \geq S_{k-1} : |z_{k-1}(t)| \geq \eta_\epsilon\}, \quad k \geq 1 \end{aligned} \quad (2.19)$$

We will see that, under the condition (2.18), both $|z_k(S_k)|$ and $|v_k(T_{k+1})|$ are $o(\eta_\epsilon)$ with large probability, then

$$2k\pi - x(S_k) = \Theta(\eta_\epsilon) \quad \text{and} \quad |x(T_{k+1}) - 2k\pi| = \Theta(\eta_\epsilon)$$

with large probability, since $v_k(S_k) = |z_k(T_{k+1})| = \eta_\epsilon$ and $x(\cdot) - 2k\pi = [z_k(\cdot) - v_k(\cdot)]/(\lambda_+ - \lambda_-)$.

At time S_k the fundamental variable is $z_k(S_k)$, since, as we showed before, $|z_k(S_k)|$ measures the distance from the k -th deterministic heteroclinic path before the criticality.

At time T_{k+1} the fundamental variable is $v_k(T_{k+1})$ since $|v_k(T_{k+1})|$ is a measure of the distance from the $k+1$ -th heteroclinic path, or, if the well has not been crossed, a measure of the distance from a suitable locked deterministic path.

$\text{sign}(z_k(T_{k+1}))$ establishes whether the solution has crossed the k -th criticality or not. If $z_k(T_{k+1}) > 0$ the k -th criticality has been passed by, if $z_k(T_{k+1}) < 0$ the solution has been trapped in the $k-1$ -th well.

The Result

We investigate the probability law of the number of wells crossed by $(x(t), p(t))$, thus we define the random variable

$$\mathcal{N} := \inf\{k \geq 0 : z_k(T_{k+1}) < 0\} \in \mathbb{N} \cup \{0\}$$

We look at the process stopped at time $T_{\mathcal{N}+1}$, $(x(t \wedge T_{\mathcal{N}+1}), p(t \wedge T_{\mathcal{N}+1}))$. All the processes labeled by k defined in the previous Sections (e.g. $(z_k(t), v_k(t))$) are well defined for $k \leq \mathcal{N}$. S_k is defined for $k \leq \mathcal{N}$ and T_k for $k \leq \mathcal{N}+1$.

We denote by \mathbf{P} the probability law of $(x(t \wedge T_{\mathcal{N}+1}), p(t \wedge T_{\mathcal{N}+1}))$, the main result is given by the following Theorem.

Theorem 2.1. *There exists $c > 0$ such that*

$$\left(\frac{1}{2} - c\epsilon^\theta\right)^{k+1} \leq \mathbf{P}\{\mathcal{N} = k\} \leq \left(\frac{1}{2} + c\epsilon^\theta\right)^{k+1} \quad (2.20)$$

for any $k \in \mathbb{N} \cup \{0\}$ and ϵ small enough.

Corollary 2.2. *From Theorem 2.1 it follows that the r.v. $\mathcal{N}+1$ has, in the limit as $\epsilon \rightarrow 0$, a geometric distribution of parameter $1/2$, i.e.*

$$\lim_{\epsilon \rightarrow 0} \mathbf{P} \{ \mathcal{N} = k \} = \frac{1}{2^{k+1}} \quad (2.21)$$

This implies, in particular, that the process crosses a finite number of wells:

$$\lim_{\epsilon \rightarrow 0} \mathbf{P} \{ \mathcal{N} < \infty \} = 1 \quad (2.22)$$

\mathcal{N} is the number of wells crossed by the process, in the sense that the first well our process is trapped is the \mathcal{N} -th one, i.e. the one where the potential has a local minimum at $(2\mathcal{N} - 1)\pi + 2x_\alpha$.

Proposition 2.3. *There exists $c > 0$ such that*

$$\lim_{\epsilon \rightarrow 0} \mathbf{P} \left\{ x(T_{\mathcal{N}+1}) \leq 2\mathcal{N}\pi - \frac{\eta_\epsilon}{(\lambda_+ - \lambda_-)} (1 - o(1)), \quad p(T_{\mathcal{N}+1}) < -\frac{\lambda_+ \eta_\epsilon}{(\lambda_+ - \lambda_-)} (1 - o(1)) \right\} = 1. \quad (2.23)$$

Once $(x(t), p(t))$ is trapped in the \mathcal{N} -well in the sense of Proposition 2.3, we expect that it remains confined in it for a long time. It is quite reasonable to think that, due to the stochastic fluctuations, the process will leave the well in an exponential time scale (order of $e^{\epsilon^{-1}}$). This is not a problem we are concerned with, since, at this level, we are looking at a faster time scale, indeed, as we will see, the time required to get across a criticality is of order $\ln(\epsilon^{-1})$. We say, thus, that the process is “confined in the \mathcal{N} -th well” in the sense that $x(t) \in (2(\mathcal{N} - 1)\pi, 2\mathcal{N}\pi)$, after $T_{\mathcal{N}+1}$, for a time that is very long if compared with $T_{\mathcal{N}+1}$ itself. For this reason it is quite reasonable to stop the process at time $T_{\mathcal{N}+1}$.

3 Definitions

Dynamics near the criticalities

For S_k as in (2.19), the process $(z_k(t), v_k(t))$ defined in (2.14) satisfies the problem

$$\begin{cases} \dot{z} := \lambda_+ z + \psi_k(x(t)) + \epsilon \dot{w} & z_k(S_k) = p(S_k) - \lambda_- (x(S_k) - 2k\pi) \\ \dot{v} := \lambda_- v + \psi_k(x(t)) + \epsilon \dot{w} & v_k(S_k) = \eta_\epsilon \end{cases}$$

with

$$\psi_k(x) = -V'(x) - \beta(x - 2k\pi) = \mathcal{O}((x - 2k\pi)^2) \quad \text{as } x \rightarrow 2k\pi$$

We denote by $(\bar{z}_k(t), \bar{v}_k(t))$ the solution of the related linear problem (2.15) starting from the same point: $(z_k(S_k), \eta_\epsilon)$. Thus, for $t \geq S_k$,

$$\bar{z}_k(t) = z_k(S_k) e^{\lambda_+(t-S_k)} + \epsilon e^{\lambda_+ t} \int_{S_k}^t e^{-\lambda_+ s} dw_s \quad (3.1)$$

and

$$\bar{v}_k(t) = \eta_\epsilon e^{\lambda_-(t-S_k)} + \epsilon e^{\lambda_- t} \int_{S_k}^t e^{-\lambda_- s} dw_s. \quad (3.2)$$

We define the errors

$$\mathcal{V}_k(t) := v_k(t) - \bar{v}_k(t) \quad \text{and} \quad \mathcal{Z}_k(t) := z_k(t) - \bar{z}_k(t) \quad (3.3)$$

thus

$$\mathcal{V}_k(t) = e^{\lambda_- t} \int_{S_k}^t e^{-\lambda_- s} \psi_k(x(s)) ds \quad \text{and} \quad \mathcal{Z}_k(t) = e^{\lambda_+ t} \int_{S_k}^t e^{-\lambda_+ s} \psi_k(x(s)) ds \quad (3.4)$$

Dynamics far from criticalities

Suppose that $T_k < \infty$ and that the solution has crossed the $k - 1$ -th criticality, then we denote by $(X_k(t), P_k(t)), t \geq T_k$ the deterministic path starting in T_k from $(x(T_k), p(T_k))$, i.e. the solution of the problem

$$\begin{cases} \dot{X} = P & X_k(T_k) = x(T_k) \\ \dot{P} = -\gamma P - V'(X) & P_k(T_k) = p(T_k) \end{cases} \quad (3.5)$$

then we expect that, as long as $x(t)$ is far enough from the criticalities, the dynamics is concentrated in a neighborhood of $(X_k(t), P_k(t))$. We denote by

$$y_k(t) := x(t) - X_k(t) \quad \text{and} \quad q_k(t) := p(t) - P_k(t)$$

thus $(y_k(t), q_k(t))$ is solution of

$$\begin{cases} \dot{y} = q & y_k(T_k) = 0 \\ \dot{q} = -\gamma q - V''(X_t)y + \varphi(X_k(t), y) + \epsilon \dot{w} & q_k(T_k) = 0 \end{cases} \quad (3.6)$$

with

$$\varphi(X, y) := V''(X)y - [V'(X + y) - V'(X)] = \mathcal{O}(y^2), \quad \text{as } |y| \rightarrow 0 \quad (3.7)$$

Assuming, for the moment, that it is possible, we define $\wp_k(x)$ as the curve on the phase plane associated to $(X_k(t), P_k(t)), t \in [T_k, S_k]$, i.e. the function such that $\wp_k(X_k(t)) = P_k(t)$ for any $t \in [T_k, S_k]$. We know from (2.5) that $\wp_k(x)$ verifies the equation

$$\wp''(x)\wp(x) = -(\wp'(x))^2 - \gamma\wp'(x) - V''(x), \quad (3.8)$$

that is obtained by deriving (2.5). Let us define the function

$$\omega_k(t) := \frac{d}{dx} \wp_k(X_k(t)) \quad (3.9)$$

then, by using (3.8), we deduce that $\omega_k(t)$ verifies the equation

$$\dot{\omega} = -\omega^2 - \gamma\omega - V''(X_k(t)). \quad (3.10)$$

It turns out to be particularly convenient to pass to the variables $(y_k(t), r_k(t))$, with

$$r_k(t) := q_k(t) - \omega_k(t)y_k(t) \quad t \geq T_k. \quad (3.11)$$

thus $(y_k(t), r_k(t))$ is solution of the problem

$$\begin{cases} \dot{y} = r + \omega_k(t)y & y_k(T_k) = 0 \\ \dot{r} = -(\gamma + \omega_k(t))r + \varphi(X_k(t), y_k(t)) + \epsilon \dot{w} & r_k(T_k) = 0 \end{cases} \quad (3.12)$$

with $\varphi(X, y)$ as above. We denote by $(\bar{y}_k(t), \bar{r}_k(t))$ the solution of the associated linear problem:

$$\begin{cases} \dot{y} = r + \omega_k(t)y & \bar{y}_k(T_k) = 0 \\ \dot{r} = -(\gamma + \omega_k(t))r + \epsilon \dot{w} & \bar{r}_k(T_k) = 0, \end{cases} \quad (3.13)$$

The convenience of this change of variables lies in the fact that the second equation in (3.13) can be solved autonomously, and then $\bar{r}_k(t)$ can be made explicit as a function of $\omega_k(t)$. We have

$$\bar{r}_k(t) = \epsilon \int_{T_k}^t e^{-\int_s^t (\omega_k(s') + \gamma) ds'} dw_s \quad (3.14)$$

Through (3.14) an explicit formula can be found also for $\bar{y}_k(t)$ as a function of $\omega_k(t)$,

$$\bar{y}_k(t) = \int_{T_k}^t \bar{r}_k(s) e^{\int_s^t \omega_k(s') ds'} ds = \epsilon \int_{T_k}^t \left(\int_s^t e^{-\int_s^{s'} (2\omega_k(s'') + \gamma) ds''} ds' \right) dw_s. \quad (3.15)$$

Finally we define the errors

$$\mathcal{Y}_k(t) := y_k(t) - \bar{y}_k(t) \quad \text{and} \quad \mathcal{R}_k(t) := r_k(t) - \bar{r}_k(t)$$

then

$$\mathcal{Y}_k(t) = \int_{T_k}^t e^{\int_s^t \omega_k(s'') ds''} \left(\int_s^t e^{-\int_s^{s'} (2\omega_k(s'') + \gamma) ds''} ds' \right) \varphi(X_k(s), y_k(s)) ds \quad (3.16)$$

and

$$\mathcal{R}_k(t) = \int_{T_k}^t e^{-\int_s^t (\omega_k(s') + \gamma) ds'} \varphi(X_k(s), y_k(s)) ds. \quad (3.17)$$

Remarks

As long as $|y_k(t)|$ is small enough, thus far from the criticalities,

$$r_k(t) = p(t) - \wp_k(X_k(t)) - \frac{d}{dx} \wp_k(X_k(t)) y_k(t) = p(t) - \wp_k(x(t)) + \mathcal{O}(y_k^2(t))$$

and, since we expect $\wp_k(x)$ to be close enough to the k -th heteroclinic orbit $\wp_k^*(x)$, we have

$$r_k(t) \simeq p(t) - \wp_k^*(x(t)) \quad (3.18)$$

The choice of the variables $(y_k(t), r_k(t))$, thus, turns out to be particularly advantageous. Indeed, with a good choice of the parameter η_ϵ , $r_k(t)$ is not far from $z_k(t)$ just before the criticality and from $v_k(t)$ just after it, i.e.

$$r_k(S_k) \simeq z_k(S_k) \quad \text{and} \quad r_{k+1}(T_{k+1}) \simeq v_k(T_{k+1})$$

as it is clear from (2.16), (2.17) and (3.18). Under this change of variables, the dynamics far from criticalities becomes “almost unidimensional”. Getting in the k -th criticality we just need, as input, the distribution of $z_k(S_k)$ that is provided, unless small errors, by $r(S_k)$. Departing from the k -th criticality we get as output the distribution of $v_k(T_{k+1}) | \{z_k(T_{k+1}) > 0\}$ that provides the approximated value of $r_{k+1}(T_{k+1})$. Away from the criticalities, the fundamental variable is thus $r_k(t)$ and we can neglect to carefully analyse the behavior of $y_k(t)$. Since the linearization of $r_k(t)$ has a quite simple form as a function of $\omega_k(t) = \wp_k(X_k(t))$ (see (3.14)) everything can be computed with a good accuracy.

We introduce some of the parameters involved. Assuming that the “ $k - 1$ -th criticality has been crossed”, we will prove in Section 4 that for any k , $\bar{z}_k(S_k) | S_k, T_k$ is a Gaussian r.v. of standard deviation $\Theta(\sigma_\epsilon)$ and expected value that is a $\mathcal{O}(\bar{\sigma}_\epsilon)$, with

$$\sigma_\epsilon := \epsilon \eta_\epsilon^{-\frac{1}{1+\theta}} = \epsilon^{1-\frac{\nu}{\theta+1}} \quad \text{and} \quad \tilde{\sigma}_\epsilon := \bar{\sigma}_\epsilon \eta_\epsilon^{\frac{\gamma}{\beta} \sqrt{\gamma^2 + 4\beta}} = \sigma_\epsilon \epsilon^\theta. \quad (3.19)$$

On the other hand, we will see that $\bar{v}_k(T_{k+1}) | S_k$ in both cases (the k -th criticality has been/not been crossed) is, for any k , a Gaussian r.v. of standard deviation $\Theta(\epsilon)$ and expected value that is a $\mathcal{O}(\bar{\sigma}_\epsilon)$ with

$$\bar{\sigma}_\epsilon := \sigma_\epsilon^{1+\theta} \eta_\epsilon^{-\theta} = \sigma_\epsilon \epsilon^{(1-\frac{2+\theta}{1+\theta} \nu)\theta} \quad (3.20)$$

It is easy to check that, under the condition (2.18) the following asymptotic relations hold that will be fundamental in our proof

$$\sigma_\epsilon = o(1), \quad \bar{\sigma}_\epsilon = o(\sigma_\epsilon) \quad \text{and} \quad \tilde{\sigma}_\epsilon = o(\bar{\sigma}_\epsilon) \quad \text{as} \quad \epsilon \rightarrow 0 \quad (3.21)$$

and

$$\epsilon = o(\eta_\epsilon^2), \quad \sigma_\epsilon = o(\eta_\epsilon) \quad \text{and} \quad \eta_\epsilon^2 = o(\tilde{\sigma}_\epsilon) \quad \text{as} \quad \epsilon \rightarrow 0 \quad (3.22)$$

The stable region linearization (2.13) is helpful as long as the dispersion around the heteroclinic solution is smaller than the distance $2k\pi - x(t)$ itself, then we impose $|\bar{z}_k(S_k)| < 2k\pi - x(S_k) = \Theta(\eta_\epsilon)$, this provides the condition $\sigma_\epsilon = o(\eta_\epsilon)$ and then the upper bound in (2.18).

On the other hand, the errors due to the linearization in the critical interval (2.13) are of order $(x(t) - 2k\pi)^2 = \Theta(\eta_\epsilon^2)$. We want such an error to be small compared with the minimum distance from the heteroclinic solution, then we impose $(x(t) - 2k\pi)^2 < |\bar{v}_k(t)| \wedge |\bar{z}_k(t)|$ for $t \in [S_k, T_{k+1}]$. This yields $\eta_\epsilon^2 = o(\tilde{\sigma}_\epsilon)$ and thus the lower bound in (2.18).

4 Proof of the main result

In this Section we prove Theorem 2.1 assuming the estimates of the errors due to the linearizations obtained in Section 7 and the estimates of the variances computed in Section 6.

Lemma 4.1. *For any fixed k , $t \geq 0$,*

$$\bar{v}_k(S_k + t) \sim \mathbf{Gauss}(\mu_v(t), \sigma_v(t))$$

with

$$\mu_v(t) = \eta_\epsilon e^{\lambda_- t}, \quad \text{and} \quad \sigma_v^2(t) = \frac{\epsilon^2}{2|\lambda_-|} (1 - e^{2\lambda_- t}) \quad (4.1)$$

$\bar{v}_k(S_k + t)$, $t \geq 0$ is independent of $\sigma(S_k)$.

Proof. Recall the definition of $\bar{v}_k(t)$ in (3.2) then it is clear that $\bar{v}_k(S_k + t)|S_k$, $t \geq 0$ has a Gaussian probability law whose average and variance are given by

$$\mathbf{E}[\bar{v}_k(S_k + t)|S_k] = \eta_\epsilon e^{\lambda_- t}$$

and

$$\begin{aligned} \mathbf{var}[\bar{v}_k(S_k + t)|S_k] &= \mathbf{E} \left[\left(\bar{v}_k(S_k + t) - \mathbf{E}[\bar{v}_k(S_k + t)|S_k] \right)^2 \middle| S_k \right] \\ &= \epsilon^2 \mathbf{E} \left[e^{2\lambda_- (S_k + t)} \left(\int_{S_k}^{S_k + t} e^{-\lambda_- s} dw_s \right)^2 \middle| S_k \right] \\ &= \epsilon^2 \mathbf{E} \left[e^{2\lambda_- (S_k + t)} \int_{S_k}^{S_k + t} e^{-2\lambda_- s} ds \right] = \frac{\epsilon^2}{2|\lambda_-|} (1 - e^{2\lambda_- t}), \end{aligned}$$

then follows the result. \square

Lemma 4.2. *We have*

$$\bar{z}_k(S_k + t) - \bar{z}_k(S_k) e^{\lambda_+ t} \sim \mathbf{Gauss}(0, \sigma_z(t)), \quad \sigma_z^2(t) := \frac{\epsilon^2}{2\lambda_+} (e^{2\lambda_+ t} - 1). \quad (4.2)$$

Proof. See the proof of Lemma 4.1. □

Let us define the process

$$\begin{aligned} \hat{z}_k(t) &:= \hat{z}_k(S_k) e^{\lambda_+(t-S_k)} + \epsilon e^{\lambda_+ t} \int_{S_k}^t e^{-\lambda_+ s} dw_s, \\ \text{with } \hat{z}_k(S_k) &:= P_k(S_k) - \wp_k^*(X_k(S_k)) + \bar{r}_k(S_k) \end{aligned} \quad (4.3)$$

and

$$\mathcal{E}_k(t) := \mathcal{Z}_k(t) + (z_k(S_k) - \hat{z}_k(S_k)) e^{\lambda_+(t-S_k)} \quad (4.4)$$

then, by (3.1), (3.3) and (4.4)

$$[z_k(t) \mid z_{k-1}(T_k) > 0] \sim \hat{z}_k(t) + \mathcal{E}_k(t), \quad t \geq S_k. \quad (4.5)$$

Lemma 4.3. For $t \geq 0$,

$$\hat{z}_k(S_k + t) \mid \hat{z}_k(S_k) = z \sim \mathbf{Gauss} \left(\mu_z(z, t), \sigma_z(t) \right)$$

with

$$\mu_z(z, t) = z e^{\lambda_+ t}, \quad \sigma_z^2(t) \quad \text{as in} \quad (4.2). \quad (4.6)$$

Proof. See the proof of Lemma 4.1. □

Lemma 4.4. We have

$$\hat{z}_k(S_k) \mid S_k, T_k \sim \mathbf{Gauss} \left(P_k(S_k) - \wp_k^*(X_k(S_k)), \sigma_r^2(T_k, S_k) \right) \quad (4.7)$$

with

$$\sigma_r^2(T_k, S_k) := \epsilon^2 \int_{T_k}^{S_k} e^{-2 \int_s^t (\omega_k(s') + \gamma) ds'} ds, \quad (4.8)$$

$\omega_k(t)$ as in (3.9).

Proof. From the definitions of $\hat{z}_k(S_k)$ and $\bar{r}_k(t)$ in (4.3) and (3.14), we have

$$\hat{z}_k(S_k) = P_k(S_k) - \wp_k^*(X_k(S_k)) + \epsilon \int_{T_k}^{S_k} e^{-\int_s^{S_k} (\omega_k(s') + \gamma) ds'} dw_s$$

thus (4.7) follows. □

Lemma 4.5. We have

$$\hat{z}_k(S_k + t) \mid S_k, T_k \sim \mathbf{Gauss} \left(\mu_{z,k}(t), \sigma_{z,k}(t) \right)$$

with

$$\mu_{z,k}(t) = e^{\lambda_+ t} (P_k(S_k) - \wp_k^*(X_k(S_k))) \quad (4.9)$$

and

$$\sigma_{z,k}^2(t) = \sigma_r^2(T_k, S_k) e^{2\lambda_+ t} + \sigma_z^2(t), \quad (4.10)$$

$\sigma_z^2(t)$ as in (4.6).

Proof. From the definition of $\hat{z}_k(t)$ and by Lemma 4.4, we know that $\hat{z}_k(S_k + t)$ is the sum of two processes that, given S_k, T_k have a Gaussian probability law, thus also $\hat{z}_k(S_k + t) | S_k, T_k$ has a Gaussian law. We have

$$\mathbf{E} [\hat{z}_k(S_k + t) | S_k, T_k] = \mathbf{E} [\hat{z}_k(S_k) | S_k, T_k] e^{\lambda_+ t}$$

that yields (4.9). By the Ito's formula

$$\hat{z}_k^2(S_k + t) = \hat{z}_k^2(S_k) + \int_{S_k}^{S_k+t} (2\lambda_+ \hat{z}_k^2(s) + \epsilon^2) ds + 2\epsilon \int_{S_k}^{S_k+t} \hat{z}_k(s) dw_s$$

thus the function $f(t) := \mathbf{E} [\hat{z}_k^2(S_k + t) | S_k, T_k]$ satisfies the equation

$$\frac{d}{dt} f(t) = 2\lambda_+ f(t) + \epsilon^2,$$

then

$$\begin{aligned} \text{var} [\hat{z}_k(S_k + t) | S_k, T_k] &= \mathbf{E} [\hat{z}_k^2(S_k + t) | S_k, T_k] \\ &= \mathbf{E} [\hat{z}_k^2(S_k) | S_k, T_k] e^{2\lambda_+ t} + \frac{\epsilon^2}{2\lambda_+} (e^{2\lambda_+ t} - 1) \end{aligned}$$

hence (4.10). □

For any $\xi > 0$ small enough, we define the sets

$$\mathcal{K}_k^\xi := \{(x, p) : p - \lambda_+(x - 2k\pi) = \eta_\epsilon, |p - \lambda_-(x - 2k\pi)| \leq \sigma_\epsilon \epsilon^{-\xi}\}. \quad (4.11)$$

and

$$\begin{aligned} \mathcal{H}_k^\xi &:= \{(x, p) : p = \eta_\epsilon + \lambda_-(x - 2(k-1)\pi), |p - \lambda_+(x - 2(k-1)\pi)| \leq \bar{\sigma}_\epsilon \epsilon^{-\xi}\}, k \geq 1 \\ \mathcal{H}_0 &:= \{(-\pi, \wp_0^*(-\pi))\} \end{aligned} \quad (4.12)$$

In the following propositions we provide some estimates on expected value and variance of $\hat{z}_k(S_k)$.

Proposition 4.6. *There exists $C > 0$ such that, for any $\zeta > 0$, ξ, ϵ small enough,*

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \{\sigma_r(T_k, S_k) = \Theta(\sigma_\epsilon)\} \geq 1 - e^{-C\epsilon^{-2\zeta}} \quad (4.13)$$

Proposition 4.7. Suppose that $(x_k, p_k) \in \mathcal{H}_k^\xi$ with ξ small enough,

$$\frac{|P_k(S_k) - \wp_k^*(X_k(S_k))|}{\sigma_r(T_k, S_k)} = \mathcal{O}(\epsilon^\theta) \quad (4.14)$$

Moreover there exists $C > 0$ such that, for any $\zeta > 0$, ξ, ϵ small enough,

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \{ |P_k(S_k) - \wp_k^*(X_k(S_k))| = \mathcal{O}(\tilde{\sigma}_\epsilon) \} \geq 1 - e^{-C\epsilon^{-2\zeta}} \quad (4.15)$$

Propositions 4.7 and 4.6 are proved in Section 7.

In the following two propositions we give some estimates on the errors due to the linearizations. In Proposition 6.15 we provide an estimate from below of $T_{k+1} - S_k$.

Proposition 4.8. There exists $C > 0$ such that, for any ζ, ϵ small enough,

$$\mathbf{1}_{(\bar{x}_k, \bar{p}_k) \in \mathcal{K}_k^\xi} \mathbf{P}_{S_k, \bar{x}_k, \bar{p}_k} \left\{ \sup_{S_k \leq t \leq T_{k+1}} |\mathcal{Z}_k(t)| e^{-\lambda_+(t-S_k)} = \mathcal{O}(\eta_\epsilon^2) \right\} \geq 1 - e^{-C\epsilon^{-2\zeta}} \quad (4.16)$$

and

$$\mathbf{1}_{(\bar{x}_k, \bar{p}_k) \in \mathcal{K}_k^\xi} \mathbf{P}_{S_k, \bar{x}_k, \bar{p}_k} \left\{ \sup_{S_k \leq t \leq T_{k+1}} |\mathcal{V}_k(t)| = \mathcal{O}(\eta_\epsilon^2) \right\} \geq 1 - e^{-C\epsilon^{-2\zeta}} \quad (4.17)$$

Proposition 4.9. There exists $C > 0$ such that, for any $\xi > 0$, ζ, ϵ small enough,

$$\mathbf{1}_{(\bar{x}_k, \bar{p}_k) \in \mathcal{K}_k^\xi} \mathbf{P}_{S_k, \bar{x}_k, \bar{p}_k} \left\{ T_{k+1} \geq S_k + \frac{1}{\lambda_+} \ln \left(\frac{\eta_\epsilon}{\sigma_\epsilon} \epsilon^\xi \right) \right\} \geq 1 - e^{-C\epsilon^{-2\zeta}} \quad (4.18)$$

We will prove Propositions 4.10, 4.8 and 4.9 in Section 7.

Proposition 4.10. There exists $C > 0$ such that, for any $\xi > 0$, ζ, ϵ small enough,

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ |z_k(S_k) - \hat{z}_k(S_k)| = \mathcal{O}(\eta_\epsilon^2 \vee \sigma_\epsilon \eta_\epsilon^{-2} \epsilon^{1-2\zeta}) \right\} \geq 1 - e^{-C\epsilon^{-2\zeta}} \quad (4.19)$$

and

$$\mathbf{1}_{(\bar{x}_{k-1}, \bar{p}_{k-1}) \in \mathcal{K}_{k-1}^\xi} \mathbf{P}_{S_{k-1}, \bar{x}_{k-1}, \bar{p}_{k-1}} \left\{ |p(T_k) - \wp_k^*(x(T_k)) - \bar{v}_{k-1}(T_k)| = \mathcal{O}(\eta_\epsilon^2) \right\} \geq 1 - e^{-C\epsilon^{-2\xi}} \quad (4.20)$$

We will denote by $\Phi(x)$ the function defined by

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-\frac{u^2}{2}} du \quad (4.21)$$

Lemma 4.11. *There exists $C > 0$ such that, for any ξ, ϵ small enough,*

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \{ |\hat{z}_k(S_k)| \leq \sigma_\epsilon \epsilon^{-\xi} \} \geq 1 - e^{-C\epsilon^{-2\xi}} \quad (4.22)$$

Proof. Let us suppose $(x_k, p_k) \in \mathcal{H}_k^\xi$ and consider the event

$$\mathbf{A}_k := \left\{ \sigma_r(T_k, S_k) = \Theta(\sigma_\epsilon), \quad |P_k(S_k) - \wp_k^*(X_k(S_k))| = \mathcal{O}(\tilde{\sigma}_\epsilon) \right\}$$

for suitable $c, c', c'' > 0$, then

$$\begin{aligned} \mathbf{P}_{T_k, x_k, p_k} \{ |\hat{z}_k(S_k)| \geq \sigma_\epsilon \epsilon^{-\xi} \} &= \mathbf{E}_{T_k, x_k, p_k} \left[\mathbf{P} \left\{ |\hat{z}_k(S_k)| \geq \sigma_\epsilon \epsilon^{-\xi} \mid S_k \right\} \right] \\ &\leq \mathbf{E}_{T_k, x_k, p_k} \left[\mathbf{1}_{\mathbf{A}_k} \mathbf{P} \left\{ |\hat{z}_k(S_k)| \geq \sigma_\epsilon \epsilon^{-\xi} \mid S_k \right\} \right] + \mathbf{P}_{T_k, x_k, p_k} \{ \mathbf{A}_k^c \} \end{aligned} \quad (4.23)$$

By Propositions 4.7 and 4.6,

$$\mathbf{P}_{T_k, x_k, p_k} \{ \mathbf{A}_k^c \} \leq e^{-C\epsilon^{-2\xi}} \quad (4.24)$$

for a suitable $C > 0$, ξ small enough, and, by Lemma 4.5, for any given T_k ,

$$\begin{aligned} \mathbf{P} \left\{ |\hat{z}_k(S_k)| \geq \sigma_\epsilon \epsilon^{-\xi} \mid S_k \right\} &= \Phi \left(\frac{\sigma_\epsilon \epsilon^{-\xi} - [P_k(S_k) - \wp_k^*(X_k(S_k))]}{\sigma_r(T_k, S_k)} \right) \\ &\quad + \Phi \left(\frac{\sigma_\epsilon \epsilon^{-\xi} - [P_k(S_k) + \wp_k^*(X_k(S_k))]}{\sigma_r(T_k, S_k)} \right) \end{aligned}$$

where

$$\mathbf{1}_{\mathbf{A}_k} \frac{\sigma_\epsilon \epsilon^{-\xi} \pm [P_k(S_k) - \wp_k^*(X_k(S_k))]}{\sigma_r(T_k, S_k)} \geq c \epsilon^{-\xi} - c' \frac{\tilde{\sigma}_\epsilon}{\sigma_\epsilon} \geq c'' \epsilon^{-\xi}$$

for suitable $c, c', c'' > 0$, since $\tilde{\sigma}_\epsilon / \sigma_\epsilon = \epsilon^\theta = o(1)$ as $\epsilon \rightarrow 0$. We have, thus,

$$\mathbf{1}_{\mathbf{A}_k} \mathbf{P} \left\{ |\hat{z}_k(S_k)| \geq \sigma_\epsilon \epsilon^{-\xi} \mid S_k \right\} \leq 2\Phi(c'' \epsilon^{-\xi}) \leq e^{-C\epsilon^{-2\xi}} \quad (4.25)$$

for some $C > 0$, then the result follows from (4.23), (4.24) and (4.25). \square

Proposition 4.12. *There exists $C > 0$ such that, for any for any ξ, ϵ small enough,*

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ (x(S_k), p(S_k)) \in \mathcal{K}_k^\xi \right\} \geq 1 - e^{-C\epsilon^{-2\xi}} \quad (4.26)$$

Proof. (4.26) follows directly from Lemma 4.11 and (4.19), since, by (3.22), $\epsilon = o(\sigma_\epsilon)$. \square

Proposition 4.13. *There exists $C > 0$ such that, for any $\epsilon, \xi > 0$ small enough,*

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ \sup_{S_k \leq t \leq T_{k+1}} |\mathcal{E}_k(t)| e^{-\lambda_+(t-S_k)} = \mathcal{O}(\eta_\epsilon^2 \vee \sigma_\epsilon \eta_\epsilon^{-2} \epsilon^{1-2\xi}) \right\} \geq 1 - e^{-C\epsilon^{-2\xi}} \quad (4.27)$$

Proof. Let us suppose $(x_k, p_k) \in \mathcal{H}_k^\xi$ and define $\mathbf{B}_k := \left\{ \sup_{S_k \leq t \leq T_{k+1}} |\mathcal{Z}_k(t)| e^{-\lambda_+(t-S_k)} = \mathcal{O}(\eta_\epsilon^2) \right\}$, we have

$$\begin{aligned}
& |\mathbf{P}_{T_k, x_k, p_k} \{ \mathbf{B}_k^c \} - \mathbf{P} \{ \mathbf{B}_k^c \mid (x(S_k), p(S_k)) \in \mathcal{K}_k^\xi \}| \\
& \leq |\mathbf{P}_{T_k, x_k, p_k} \{ \mathbf{B}_k^c, (x(S_k), p(S_k)) \in \mathcal{K}_k^\xi \} - \mathbf{P} \{ \mathbf{B}_k^c \mid (x(S_k), p(S_k)) \in \mathcal{K}_k^\xi \}| \\
& \quad + \mathbf{P}_{T_k, x_k, p_k} \{ (x(S_k), p(S_k)) \notin \mathcal{K}_k^\xi \} \\
& \leq \mathbf{P} \{ \mathbf{B}_k^c \mid (x(S_k), p(S_k)) \in \mathcal{K}_k^\xi \} |1 - \mathbf{P}_{T_k, x_k, p_k} \{ (x(S_k), p(S_k)) \in \mathcal{K}_k^\xi \}| \\
& \quad + \mathbf{P}_{T_k, x_k, p_k} \{ (x(S_k), p(S_k)) \notin \mathcal{K}_k^\xi \} \\
& \leq 2\mathbf{P}_{T_k, x_k, p_k} \{ (x(S_k), p(S_k)) \notin \mathcal{K}_k^\xi \}
\end{aligned}$$

On the other hand,

$$\mathbf{P} \{ \mathbf{B}_k^c \mid (x(S_k), p(S_k)) \in \mathcal{K}_k^\xi \} \leq \sup_{(\bar{x}_k, \bar{p}_k) \in \mathcal{K}_k^\xi} \mathbf{P}_{S_k, \bar{x}_k, \bar{p}_k} \{ \mathbf{B}_k^c \} \quad (4.28)$$

then, from (4.16) and (4.26) it follows that there exists $C > 0$ such that, for ξ, ϵ small enough,

$$\mathbf{P}_{T_k, x_k, p_k} \{ \mathbf{B}_k^c \} \leq e^{-C\epsilon^{-2\xi}} \quad (4.29)$$

thus, recalling that $\mathcal{E}_k(t) e^{-\lambda_+(t-S_k)} = \mathcal{Z}_k(t) e^{-\lambda_+(t-S_k)} + (z_k(S_k) - \hat{z}_k(S_k))$, (4.27) follows directly from (4.19) and (4.29). \square

Proposition 4.14. *There exists $c > 0$ such that, for any $\epsilon > 0$ small enough,*

$$\left| \mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \{ z_k(T_{k+1}) \geq 0 \} - \frac{1}{2} \right| \leq c\epsilon^\theta.$$

Proof. We assume $(x_k, p_k) \in \mathcal{H}_k^\xi$. We define the event

$$\mathbf{C}_k := \left\{ \sup_{S_k \leq t \leq T_{k+1}} |\mathcal{E}_k(t)| e^{-\lambda_+(t-S_k)} = \mathcal{O}(v_\epsilon^\xi) \right\}. \quad (4.30)$$

with $v_\epsilon^\xi := \eta_\epsilon^2 \vee \sigma_\epsilon \eta_\epsilon^{-2} \epsilon^{1-2\xi}$. By (4.5) we have

$$\begin{aligned}
\mathbf{P}_{T_k, x_k, p_k} \{ \mathbf{C}_k, z_k(T_{k+1}) > 0 \} &= \mathbf{P}_{T_k, x_k, p_k} \{ \mathbf{C}_k, \hat{z}_k(T_{k+1}) > -\mathcal{E}_k(T_{k+1}) \} \\
&\leq \mathbf{P}_{T_k, x_k, p_k} \left\{ \hat{z}_k(T_{k+1}) > -c v_\epsilon e^{\lambda_+(T_{k+1}-S_k)} \right\}
\end{aligned}$$

For \mathbf{A}_k as in the Proof of Lemma 4.11, we have

$$\begin{aligned}
& \mathbf{P}_{T_k, x_k, p_k} \left\{ \hat{z}_k(T_{k+1}) > -c v_\epsilon e^{\lambda_+(T_{k+1}-S_k)} \right\} \\
& \leq \mathbf{E}_{T_k, x_k, p_k} \left[\mathbf{1}_{\mathbf{A}_k} \mathbf{P} \left\{ \hat{z}_k(T_{k+1}) > -c v_\epsilon e^{\lambda_+(T_{k+1}-S_k)} \mid S_k \right\} \right] \\
& \quad + \mathbf{P}_{T_k, x_k, p_k} \{ \mathbf{A}_k^c \}.
\end{aligned} \quad (4.31)$$

By Lemma 4.5, for any given T_k we have

$$\begin{aligned} & \mathbf{P} \left\{ \hat{z}_k(T_{k+1}) > -c v_\epsilon e^{\lambda_+(T_{k+1}-S_k)} \mid S_k \right\} \\ &= \Phi \left(\frac{-\mu_{z,k}(T_{k+1}-S_k) - c v_\epsilon e^{\lambda_+(T_{k+1}-S_k)}}{\sigma_{z,k}(T_{k+1}-S_k)} \right) \end{aligned} \quad (4.32)$$

where, from (4.10) and (4.9),

$$\begin{aligned} & \mathbf{1}_{\mathbf{A}_k} \frac{\mu_{z,k}(T_{k+1}-S_k) + c v_\epsilon e^{\lambda_+(T_{k+1}-S_k)}}{\sigma_{z,k}(T_{k+1}-S_k)} \\ & \leq \mathbf{1}_{\mathbf{A}_k} \frac{[P_k(S_k) - \wp_k^*(X_k(S_k))] + c v_\epsilon}{\sigma_r(T_k, S_k)} \\ & \leq c' \frac{\tilde{\sigma}_\epsilon + v_\epsilon}{\sigma_\epsilon} \leq c'' \epsilon^\theta \end{aligned} \quad (4.33)$$

for suitable $c', c'' > 0$, since, from (2.18), $v_\epsilon = o(\tilde{\sigma}_\epsilon)$. By (4.31), (4.32), (4.33) and (4.24) it follows that

$$\mathbf{P}_{T_k, x_k, p_k} \{ \mathbf{C}_k, z_k(T_{k+1}) > 0 \} \leq \Phi(-c'' \epsilon^\theta) + e^{-C \epsilon^{-2\xi}} \leq \frac{1}{2} + c''' \epsilon^\theta$$

for suitable $C, c''' > 0$, hence, by Proposition 4.13, there exist $c, c', C > 0$ such that

$$\begin{aligned} \mathbf{P}_{T_k, x_k, p_k} \{ z_k(T_{k+1}) > 0 \} & \leq \mathbf{P}_{T_k, x_k, p_k} \{ \mathbf{C}_k, z_k(T_{k+1}) > 0 \} + \mathbf{P}_{T_k, x_k, p_k} \{ \mathbf{C}_k^c \} \\ & \leq \frac{1}{2} + c \left(\epsilon^\theta + e^{-C \epsilon^{-2\xi}} \right) \leq \frac{1}{2} + c' \epsilon^\theta. \end{aligned}$$

Analogously,

$$\begin{aligned} \mathbf{P}_{T_k, x_k, p_k} \{ \mathbf{C}_k, z_k(T_{k+1}) < 0 \} &= \mathbf{P}_{T_k, x_k, p_k} \{ \mathbf{C}_k, \hat{z}_k(T_{k+1}) < -\mathcal{E}_k(T_{k+1}) \} \\ &\leq \mathbf{P}_{T_k, x_k, p_k} \left\{ \hat{z}_k(T_{k+1}) < c v_\epsilon e^{\lambda_+(T_{k+1}-S_k)} \right\} \\ &\leq \frac{1}{2} + e^{-C \epsilon^{-2\xi}} \end{aligned}$$

for some $C > 0$, hence

$$\begin{aligned} \mathbf{P}_{T_k, x_k, p_k} \{ z_k(T_{k+1}) < 0 \} & \leq \mathbf{P}_{T_k, x_k, p_k} \{ \mathbf{C}_k, z_k(T_{k+1}) < 0 \} + \mathbf{P}_{T_k, x_k, p_k} \{ \mathbf{C}_k^c \} \\ & \leq \frac{1}{2} + e^{-C' \epsilon^{-2\xi}} \end{aligned}$$

for some $C' > 0$, this concludes the proof of the Proposition. \square

Lemma 4.15. *There exists $C > 0$ such that, for any fixed $\epsilon, \xi > 0$ small enough,*

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ T_{k+1} \leq S_k + \frac{1}{\lambda_+} \ln \left(\frac{\eta_\epsilon}{\sigma_\epsilon} \epsilon^{-\xi} \right) \right\} \geq 1 - C \epsilon^\xi \quad (4.34)$$

Proof. We assume $(x_k, p_k) \in \mathcal{H}_k^\xi$ for some ξ small enough. Let \mathbf{C}_k be as in (4.30) and define $\tau := \ln(\eta_\epsilon \sigma_\epsilon^{-1} \epsilon^{-\xi}) / \lambda_+$, we have

$$\begin{aligned} \mathbf{P}_{T_k, x_k, p_k} \{ T_{k+1} \geq S_k + \tau \} &= \mathbf{P}_{T_k, x_k, p_k} \left\{ \sup_{0 \leq t \leq \tau} |z_k(S_k + t)| \leq \eta_\epsilon \right\} \\ &\leq \mathbf{P}_{T_k, x_k, p_k} \left\{ \sup_{0 \leq t \leq \tau} |z_k(S_k + t)| \leq \eta_\epsilon, \mathbf{C}_k \right\} + \mathbf{P}_{T_k, x_k, p_k} \{ \mathbf{C}_k^c \} \end{aligned}$$

By (4.27) we know that there exists $C > 0$ such that $\mathbf{P}_{T_k, x_k, p_k} \{\mathbf{C}_k^c\} \leq e^{-C\epsilon^{-2\xi}}$ for any ϵ, ξ small enough. From the conditions in (3.22) we have $(\eta_\epsilon^2 \vee \sigma_\epsilon \eta_\epsilon^{-2} \epsilon^{1-2\xi})e^{\lambda_+ \tau} = o(\eta_\epsilon)$, thus, by (4.5) and the definition of \mathbf{C}_k ,

$$\begin{aligned} & \mathbf{P}_{T_k, x_k, p_k} \left\{ \sup_{0 \leq t \leq \tau} |z_k(S_k + t)| \leq \eta_\epsilon, \mathbf{C}_k \right\} \\ & \leq \mathbf{P}_{T_k, x_k, p_k} \left\{ \sup_{0 \leq t \leq \tau} |\hat{z}_k(S_k + t)| \leq 2\eta_\epsilon \right\} \\ & = \mathbf{E}_{T_k, x_k, p_k} \left[\mathbf{P} \left\{ \sup_{0 \leq t \leq \tau} |\hat{z}_k(S_k + t)| \leq 2\eta_\epsilon \mid S_k \right\} \right] \end{aligned}$$

It is easy to show from Lemma 4.3, Proposition 4.7 and Proposition 4.6 that for any given T_k there exists $C > 0$ such that, for ϵ small enough,

$$\mathbf{P} \left\{ \sup_{0 \leq t \leq \tau} |\hat{z}_k(S_k + t)| \leq 2\eta_\epsilon \mid S_k \right\} \leq C\epsilon^\xi$$

then we get (4.34). □

Corollary 4.16. *If $(x_k, p_k) \in \mathcal{H}_k^\xi$ for some $\xi > 0$ small enough then, for any fixed $\epsilon > 0$ small enough, $k \geq 0$, $T_{k+1} - S_k$ is $\mathbf{P}_{T_k, x_k, p_k}$ -a.s. finite.*

Proof. It directly follows from the previous Lemma. □

Lemma 4.17. *There exists $C > 0$ such that, for any $\epsilon, \xi > 0$ small enough,*

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ |v_k(T_{k+1})| \geq \bar{\sigma}_\epsilon \epsilon^{-\xi} \right\} \leq C\epsilon^\xi \quad (4.35)$$

Proof. Assume that $(x_k, p_k) \in \mathcal{H}_k^\xi$, we first prove that there exists $C > 0$ such that

$$\mathbf{P}_{T_k, x_k, p_k} \left\{ |\bar{v}_k(T_{k+1})| \geq \bar{\sigma}_\epsilon \epsilon^{-\xi} \right\} \leq C\epsilon^\xi \quad (4.36)$$

for any ϵ, ξ small enough.

Let us define the event $\mathbf{D}_k := \{T_{k+1} \geq S_k + \frac{1}{\lambda_+} \ln(\eta_\epsilon \epsilon^\xi \sigma_\epsilon^{-1})\}$, then

$$\begin{aligned} & \mathbf{P}_{T_k, x_k, p_k} \left\{ |\bar{v}_k(T_{k+1}) - \mu_v(T_{k+1} - S_k)| \geq \epsilon^{1-\xi} \right\} \\ & \leq \mathbf{E}_{T_k, x_k, p_k} \left[\mathbf{1}_{\mathbf{D}_k} \mathbf{P} \left\{ |\bar{v}_k(T_{k+1}) - \mu_v(T_{k+1} - S_k)| \geq \epsilon^{1-\xi} \mid S_k \right\} \right] \\ & \quad + \mathbf{P}_{T_k, x_k, p_k} \{\mathbf{D}_k^c\} \end{aligned}$$

where, by Lemma 4.1, for any fixed T_{k+1} ,

$$\begin{aligned} & \mathbf{P} \left\{ |\bar{v}_k(T_{k+1}) - \mu_v(T_{k+1} - S_k)| \geq \epsilon^{1-\xi} \mid S_k \right\} \\ & = 2\Phi \left(\frac{\epsilon^{1-\xi}}{\sigma_v(T_{k+1} - S_k)} \right) \end{aligned}$$

By (4.1) we have

$$\mathbf{1}_{\mathbf{D}_k} \Phi \left(\frac{\epsilon^{1-\xi}}{\sigma_v(T_{k+1} - S_k)} \right) \leq \Phi(c\epsilon^{-\xi}) \leq c'\epsilon^{-\xi} e^{-c''\epsilon^{-2\xi}}$$

for some $c, c', c'' > 0$, moreover, by Lemma (4.15) there exists C such that

$$\mathbf{P}_{T_k, x_k, p_k} \{\mathbf{D}_k^c\} \leq C\epsilon^\xi \quad (4.37)$$

hence there exists $C' > 0$ such that

$$\mathbf{P}_{T_k, \bar{x}, p_k} \left\{ |\bar{v}_k(T_{k+1}) - \mu_v(T_{k+1} - S_k)| \geq \epsilon^{1-\xi} \right\} \leq C'\epsilon^\xi \quad (4.38)$$

By (4.1), $\mu_v(T_{k+1} - S_k) = \eta_\epsilon e^{\lambda_-(T_{k+1} - S_k)}$, thus, from (4.37) and the definition of $\bar{\sigma}_\epsilon$

$$\mathbf{P}_{T_k, \bar{x}, p_k} \left\{ \mu_v(T_{k+1} - S_k) \geq \bar{\sigma}_\epsilon \epsilon^{-\xi} \right\} \leq C\epsilon^\xi \quad (4.39)$$

then (4.35) descends from (4.38) and (4.39) since $\epsilon = o(\bar{\sigma}_\epsilon)$. We recall now that $v_k(t) = \bar{v}_k(t) + \mathcal{V}_k(t)$, then (4.38) follows from (4.35) and (4.17) since $\eta_\epsilon^2 = o(\bar{\sigma}_\epsilon)$. \square

We decompose the set \mathcal{H}_k^ξ (see the definition in (4.12)) in the two subests:

$$\mathcal{H}_k^\xi = \mathcal{L}_k^\xi \cap \mathcal{M}_k \quad \text{for } k \geq 1 \quad (4.40)$$

with

$$\mathcal{L}_k^\xi := \{(x, p) : |p - \lambda_+(x - 2(k-1)\pi)| \leq \bar{\sigma}_\epsilon \epsilon^{-\xi}\} \quad (4.41)$$

and

$$\mathcal{M}_k := \{(x, p) : p = \eta_\epsilon + \lambda_-(x - 2(k-1)\pi)\} \quad (4.42)$$

and recall that $\mathcal{H}_0 = \{(-\pi, \wp_0^*(-\pi))\}$.

Lemma 4.18. *Suppose that $(x, p) \in \mathcal{H}_{k-1}$, then there exist $c, C > 0$ such that*

$$\mathbf{P}_{T_{k-1}, x, p} \left\{ (x(T_k), p(T_k)) \notin \mathcal{L}_k^\xi \right\} \leq C\epsilon^\xi \quad (4.43)$$

and

$$\left| \mathbf{P}_{T_{k-1}, x, p} \left\{ (x(T_k), p(T_k)) \in \mathcal{H}_k^\xi \right\} - \frac{1}{2} \right| \leq c\epsilon^\theta \quad (4.44)$$

for any ϵ small enough.

Proof. (4.43) follows from (4.35) since $(x(T_k), p(T_k)) \notin \mathcal{L}_k^\xi$ if and only if $|v_{k-1}(T_k)| \geq \bar{\sigma}_\epsilon \epsilon^{-\xi}$. On the other hand, from Proposition 4.14 it follows that

$$\left| \mathbf{P}_{T_{k-1}, x, p} \left\{ (x(T_k), p(T_k)) \in \mathcal{M}_k \right\} - \frac{1}{2} \right| \leq c\epsilon^\theta \quad (4.45)$$

since $(x(T_k), p(T_k)) \in \mathcal{M}_k$ if and only if $z_{k-1}(T_k) > 0$. We have

$$\begin{aligned} & \mathbf{P}_{T_{k-1}, x, p} \left\{ (x(T_k), p(T_k)) \in \mathcal{H}_k^\xi \right\} \\ &= \mathbf{P}_{T_{k-1}, x, p} \left\{ (x(T_k), p(T_k)) \in \mathcal{M}_k \right\} - \mathbf{P}_{T_{k-1}, x, p} \left\{ (x(T_k), p(T_k)) \in \mathcal{M}_k \cup (\mathcal{L}_k^\xi)^c \right\} \end{aligned}$$

then, from (4.43)

$$\begin{aligned} & \left| \mathbf{P}_{T_{k-1}, x, p} \left\{ (x(T_k), p(T_k)) \in \mathcal{H}_k^\xi \right\} - \mathbf{P}_{T_{k-1}, x, p} \left\{ (x(T_k), p(T_k)) \in \mathcal{M}_k \right\} \right| \\ & \leq \mathbf{P}_{T_{k-1}, x, p} \left\{ (x(T_k), p(T_k)) \notin \mathcal{L}_k^\xi \right\} \leq C\epsilon^\xi \end{aligned} \quad (4.46)$$

then (4.44) follows from (4.45) and (4.46) since ξ is arbitrary. \square

Lemma 4.19. *Let $(x, p) \in \mathcal{H}_0$, then there exist $c, C > 0$ such that*

$$\left(\frac{1}{2} - c\epsilon^\theta \right)^k - C\epsilon^\xi \leq \mathbf{P}_{T_0, x, p} \left\{ (x(T_k), p(T_k)) \in \mathcal{H}_k^\xi \right\} \leq \left(\frac{1}{2} + c\epsilon^\theta \right)^k + C\epsilon^\xi \quad (4.47)$$

for any ϵ small enough.

Proof. We prove the upper bound. For $(x, p) \in \mathcal{H}_{k-2}^\xi$ we have

$$\begin{aligned} & \mathbf{P}_{T_{k-2}, x, p} \left\{ (x(T_k), p(T_k)) \in \mathcal{H}_k^\xi \right\} \\ &= \mathbf{E}_{T_{k-2}, x, p} \left[\mathbf{1}_{(x, p) \in \mathcal{M}_{k-1}} \mathbf{P}_{T_{k-1}, x, p} \left\{ (x(T_k), p(T_k)) \in \mathcal{H}_k^\xi \right\} \right] \\ &\leq \mathbf{E}_{T_{k-2}, x, p} \left[\mathbf{1}_{(x, p) \in \mathcal{H}_{k-1}^\xi} \mathbf{P}_{T_{k-1}, x, p} \left\{ (x(T_k), p(T_k)) \in \mathcal{H}_k^\xi \right\} \right] + \mathbf{P}_{T_{k-2}, x, p} \left\{ (x(T_{k-1}), p(T_{k-1})) \notin \mathcal{L}_{k-1} \right\} \\ &\leq \left(\frac{1}{2} + c\epsilon^\theta \right) \mathbf{P}_{T_{k-2}, x, p} \left\{ (x(T_{k-1}), p(T_{k-1})) \in \mathcal{H}_{k-1}^\xi \right\} + C\epsilon^\xi \\ &\leq \left(\frac{1}{2} + c\epsilon^\theta \right)^2 + C\epsilon^\xi \end{aligned} \quad (4.48)$$

where the two last inequalities follow from (4.43) and (4.44). By repeating k times this argument we find that, if $(x, p) \in \mathcal{H}_0$,

$$\mathbf{P}_{T_0, x, p} \left\{ (x(T_k), p(T_k)) \in \mathcal{H}_k^\xi \right\} \leq \left(\frac{1}{2} + c\epsilon^\theta \right)^k + C\epsilon^\xi \sum_{i=0}^{k-2} \left(\frac{1}{2} + c\epsilon^\theta \right)^i \leq \left(\frac{1}{2} + c\epsilon^\theta \right)^k + C'\epsilon^\xi \quad (4.49)$$

The lower bound follows from the same argument. \square

Lemma 4.20. *Let $(x, p) \in \mathcal{H}_0$, then there exists $C > 0$ such that*

$$\mathbf{P}_{T_0, x, p} \left\{ (x(T_k), p(T_k)) \notin \mathcal{L}_k^\xi \right\} \leq C\epsilon^\xi \quad (4.50)$$

for any ϵ small enough.

Proof. For $(x, p) \in \mathcal{H}_{k-2}^\xi$ we have

$$\begin{aligned}
& \mathbf{P}_{T_{k-2}, x, p} \left\{ (x(T_k), p(T_k)) \notin \mathcal{L}_k^\xi \right\} = \\
& = \mathbf{E}_{T_{k-2}, x, p} \left[\mathbf{1}_{(x, p) \in \mathcal{M}_{k-1}} \mathbf{P}_{T_{k-1}, x, p} \left\{ (x(T_k), p(T_k)) \notin \mathcal{L}_k^\xi \right\} \right] \\
& \leq \mathbf{E}_{T_{k-2}, x, p} \left[\mathbf{1}_{(x, p) \in \mathcal{H}_{k-1}^\xi} \mathbf{P}_{T_{k-1}, x, p} \left\{ (x(T_k), p(T_k)) \notin \mathcal{L}_k \right\} \right] + \mathbf{P}_{T_{k-2}, x, p} \left\{ (x(T_{k-1}), p(T_{k-1})) \notin \mathcal{L}_{k-1} \right\} \\
& \leq C\epsilon^\xi \mathbf{P}_{T_{k-2}, x, p} \left\{ (x(T_{k-1}), p(T_{k-1})) \in \mathcal{H}_{k-1}^\xi \right\} + C\epsilon^\xi \\
& \leq C\epsilon^\xi \left[\left(\frac{1}{2} + c\epsilon^\theta \right) + 1 \right]
\end{aligned} \tag{4.51}$$

the last two inequalities descending from (4.43) and (4.44). By repeating k times this argument we find that, for $(x, p) \in \mathcal{H}_0$,

$$\mathbf{P}_{T_0, x, p} \left\{ (x(T_k), p(T_k)) \notin \mathcal{L}_k^\xi \right\} \leq C\epsilon^\xi \sum_{i=0}^{k-1} \left(\frac{1}{2} + c\epsilon^\theta \right)^i \leq C'\epsilon^\xi \tag{4.52}$$

this concludes the proof of the Lemma. \square

Conclusion of the Proof of Theorem 2.1. The processes $z_k(t)$ are well defined only for $k \leq \mathcal{N}$, then we set

$$z_k(t) \equiv 0 \quad \text{for} \quad k \geq \mathcal{N} + 1$$

Let $(x, p) \in \mathcal{H}_0$, then we prove that there exists $C > 0$ such that

$$\left| \mathbf{P}_{T_0, x, p} \{ \mathcal{N} = k \} - \left(\frac{1}{2} + c\epsilon^\theta \right)^{k+1} \right| \leq C\epsilon^\xi \tag{4.53}$$

for any ϵ small enough. We have

$$\begin{aligned}
& \mathbf{P}_{T_0, x, p} \{ \mathcal{N} = k \} = \mathbf{P}_{T_0, x, p} \{ z_k(T_{k+1}) < 0, z_{k-1}(T_k) > 0, \dots, z_0(T_1) > 0 \} \\
& = \mathbf{P}_{T_0, x, p} \{ z_k(T_{k+1}) < 0, z_{k-1}(T_k) > 0 \} \\
& = \mathbf{P}_{T_0, x, p} \{ (x(T_{k+1}), p(T_{k+1})) \notin \mathcal{M}_{k+1}, (x(T_k), p(T_k)) \in \mathcal{M}_k \} \\
& = \mathbf{E}_{T_0, x, p} \left[\mathbf{1}_{(x, p) \in \mathcal{M}_k} \mathbf{P}_{T_k, x, p} \{ (x(T_{k+1}), p(T_{k+1})) \notin \mathcal{M}_{k+1} \} \right] \\
& \leq \mathbf{E}_{T_0, x, p} \left[\mathbf{1}_{(x, p) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x, p} \{ (x(T_{k+1}), p(T_{k+1})) \notin \mathcal{M}_{k+1} \} \right] + \mathbf{P}_{T_0, x, p} \left\{ (x(T_k), p(T_k)) \notin \mathcal{L}_k^\xi \right\} \\
& \leq \left(\frac{1}{2} + c\epsilon^\theta \right) \mathbf{P}_{T_0, x, p} \left\{ (x(T_k), p(T_k)) \in \mathcal{H}_k^\xi \right\} + C\epsilon^\xi \\
& \leq \left(\frac{1}{2} + c\epsilon^\theta \right) \left[\left(\frac{1}{2} + c\epsilon^\theta \right)^k + C\epsilon^\xi \right] + C\epsilon^\xi \\
& \leq \left(\frac{1}{2} + c\epsilon^\theta \right)^{k+1} + C'\epsilon^\xi
\end{aligned}$$

the last three inequalities descending from (4.45), (4.50) and (4.47). This yields the upper bound in (4.53), the lower bound can be obtained by an analogous argument. Thus Theorem 2.1 follows from (4.53). \square

Proof of Proposition 2.3. For $\xi > 0$ we define the set

$$\mathcal{Q}_k^\xi := \left\{ (x, p) : x \leq 2k\pi - \frac{\eta_\epsilon}{(\lambda_+ - \lambda_-)} \left(1 - \frac{\bar{\sigma}_\epsilon}{\eta_\epsilon} \epsilon^{-\xi} \right), p < -\frac{\lambda_+ \eta_\epsilon}{(\lambda_+ - \lambda_-)} \left(1 - \frac{|\lambda_-| \bar{\sigma}_\epsilon \epsilon^{-\xi}}{\lambda_+ \eta_\epsilon} \right) \right\}$$

We prove that, for $(x, p) \in \mathcal{H}_0$,

$$\lim_{\epsilon \rightarrow 0} \mathbf{P}_{T_0, x, p} \left\{ (x(T_{\mathcal{N}+1}), p(T_{\mathcal{N}+1})) \notin \mathcal{Q}_{\mathcal{N}}^\xi \right\} = 0 \quad (4.54)$$

We have

$$\mathbf{P}_{T_0, x, p} \left\{ (x(T_{\mathcal{N}+1}), p(T_{\mathcal{N}+1})) \notin \mathcal{Q}_{\mathcal{N}}^\xi \right\} \quad (4.55)$$

$$= \mathbf{P}_{T_0, x, p} \left\{ (x(T_{\mathcal{N}+1}), p(T_{\mathcal{N}+1})) \notin \mathcal{Q}_{\mathcal{N}}^\xi, z_{\mathcal{N}}(T_{\mathcal{N}+1}) < 0, z_{\mathcal{N}-1}(T_{\mathcal{N}}) > 0 \right\} \quad (4.56)$$

$$= \mathbf{E}_{T_0, x, p} \left[\mathbf{1}_{(x, p) \in \mathcal{M}_{\mathcal{N}}} \mathbf{P}_{T_{\mathcal{N}}, x, p} \left\{ (x(T_{\mathcal{N}+1}), p(T_{\mathcal{N}+1})) \notin \mathcal{Q}_{\mathcal{N}}^\xi, z_{\mathcal{N}}(T_{\mathcal{N}+1}) = -\eta_\epsilon \right\} \right] \quad (4.57)$$

it is easy to check from the definition of \mathcal{Q}_k^ξ that

$$\left\{ (x(T_{\mathcal{N}+1}), p(T_{\mathcal{N}+1})) \notin \mathcal{Q}_{\mathcal{N}}^\xi, z_{\mathcal{N}}(T_{\mathcal{N}+1}) = -\eta_\epsilon \right\} = \left\{ |v_{\mathcal{N}}(T_{\mathcal{N}+1})| \geq \bar{\sigma}_\epsilon \epsilon^{-\xi} \right\}$$

then

$$\begin{aligned} \mathbf{P}_{T_0, x, p} \left\{ (x(T_{\mathcal{N}+1}), p(T_{\mathcal{N}+1})) \notin \mathcal{Q}_{\mathcal{N}}^\xi \right\} &= \mathbf{E}_{T_0, x, p} \left[\mathbf{1}_{(x, p) \in \mathcal{M}_{\mathcal{N}}} \mathbf{P}_{T_{\mathcal{N}}, x, p} \left\{ |v_{\mathcal{N}}(T_{\mathcal{N}+1})| \geq \bar{\sigma}_\epsilon \epsilon^{-\xi} \right\} \right] \\ &\leq \mathbf{E}_{T_0, x, p} \left[\mathbf{1}_{(x, p) \in \mathcal{H}_{\mathcal{N}}^\xi} \mathbf{P}_{T_{\mathcal{N}}, x, p} \left\{ |v_{\mathcal{N}}(T_{\mathcal{N}+1})| \geq \bar{\sigma}_\epsilon \epsilon^{-\xi} \right\} \right] + \mathbf{P}_{T_0, x, p} \left\{ (x(T_{\mathcal{N}}), p(T_{\mathcal{N}})) \notin \mathcal{L}_{\mathcal{N}}^\xi \right\} \end{aligned}$$

then (4.54) follows from (4.35), (4.50) and (2.22). \square

5 Deterministic paths

In this Section we study the qualitative behavior of the orbits of the system (3.5) lying in a neighborhood of the heteroclinic path. We recall that $\wp_k^*(x)$ is the heteroclinic orbit defined in $(2(k-1)\pi, 2k\pi)$. We have the following result.

Lemma 5.1. *Let us fix $\delta > 0$ small enough, then, for any $2(k-1)\pi < x \leq 2(k-1)\pi + \delta$,*

$$\wp_k^*(x) = \lambda_+ (1 + \mathcal{O}(\delta))(x - 2(k-1)\pi) \quad (5.1)$$

whereas, for any $2k\pi - \delta \leq x < 2k\pi$

$$\wp_k^*(x) = \lambda_- (1 + \mathcal{O}(\delta))(x - 2k\pi) \quad (5.2)$$

Proof. It follows directly from (2.11). \square

We denote by $\wp_k(x)$ a generic orbit in the phase plane close enough to $\wp_k^*(x)$ in $(2(k-1)\pi, 2k\pi)$ in the following sense. We fix $\eta > 0$ small enough and define $x_k, x'_k \in (2(k-1)\pi, 2k\pi)$, $x_k < x'_k$ such that

$$x_k - 2(k-1)\pi = \Theta(\eta) \quad \text{and} \quad 2k\pi - x'_k = \Theta(\eta) \quad \text{as} \quad \eta \rightarrow 0 \quad (5.3)$$

we suppose

$$|\wp_k(x_k) - \wp_k^*(x_k)| = o(\eta) \quad \text{as} \quad \eta \rightarrow 0 \quad (5.4)$$

then, as we will see in the following Lemma, $\wp_k(x)$ is well defined in $[x_k, x'_k]$.

Lemma 5.2. For x_k, x'_k as in (5.3) and $\wp_k(x)$ satisfying (5.4), we define $\rho_k(x) := \wp_k(x) - \wp_k^*(x)$. Let $f_k^\delta(x) := \delta^{-\frac{\beta}{\lambda_+^2}} e^{\int_\delta^x \frac{V'(u)}{\wp_k^{*2}(u)} du}$ then, for any $\delta > \eta$ small enough,

$$\rho_k(x) = \rho_k(x_k) \left(\frac{x_k}{x} \right)^{\frac{\beta}{\lambda_+^2}} (1 + \mathcal{O}(\delta)) \quad \text{for} \quad x_k \leq x \leq 2(k-1)\pi + \delta \quad (5.5)$$

$$\rho_k(x) = \rho_k(x_k) x_k^{\frac{\beta}{\lambda_+^2}} f_k^\delta(x) (1 + \mathcal{O}(\delta)) \quad \text{for} \quad 2(k-1)\pi + \delta \leq x \leq 2k\pi - \delta \quad (5.6)$$

and

$$\rho_k(x) = \rho_k(x_k) \frac{x_k^{\frac{\beta}{\lambda_+^2}}}{(2k\pi - x)^{\frac{\beta}{\lambda_-^2}}} f_k^\delta(2k\pi - \delta) (1 + \mathcal{O}(\delta)) \quad \text{for} \quad 2k\pi - \delta \leq x \leq x'_k \quad (5.7)$$

Proof. Because of the periodicity of the dynamics, it is sufficient to prove the result for $k = 1$. In order to lighten the notation, we omit the index 1 in $\wp_1(x), \wp_1^*(x)$, etc.

We define $x_* := \inf\{x \geq x_1 : |\rho(x)| \geq \eta\}$, thus, by (2.5),

$$\rho'(x) = \frac{V'(x)}{\wp^*(x)\wp(x)} = \frac{V'(x)}{\wp^*(x)(\wp^*(x) + \rho(x))} \quad (5.8)$$

then

$$\rho(x) = \rho(x_1) e^{\int_{x_1}^x \frac{V'(u)}{\wp^*(u)(\wp^*(u) + \rho(u))} du} \quad \text{for} \quad x \geq x_1 \quad (5.9)$$

hence there exists a function $g(\cdot)$ such that

$$\rho(x) = \rho(x_1) e^{(1+g(x)) \int_{x_1}^x \frac{V'(u)}{\wp^{*2}(u)} du} \quad \text{and} \quad \sup_{x_1 < x \leq x_*} |g(x)| \leq c\eta \quad (5.10)$$

for some $c > 0$. We fix $\delta > 0$ small enough, $\delta > \eta$, then, by (2.9), for any $k \geq 0$,

$$V'(x) = -\beta(x - 2k\pi)(1 + \mathcal{O}(\delta)) \quad \text{for} \quad |x - 2k\pi| \leq \delta \quad (5.11)$$

By (5.1) and (5.11), there exists $\bar{g}(x)$, with $\sup_{0 \leq x \leq \delta} |\bar{g}(x)| = \mathcal{O}(\delta)$ as $\delta \rightarrow 0$, such that

$$\int_{x_1}^x \frac{V'(u)}{\wp^{*2}(u)} du = -\frac{\beta}{\lambda_+^2} \ln \left(\frac{x}{x_1} \right) + \bar{g}(x) \quad \text{for} \quad x_1 \leq x \leq \delta \quad (5.12)$$

and, by (5.2) and (5.11) there exists $\tilde{g}(x)$, with $\sup_{2\pi - \delta \leq x \leq 2\pi} |\tilde{g}(x)| = \mathcal{O}(\delta)$ as $\delta \rightarrow 0$, such that

$$\int_{2\pi - \delta}^x \frac{V'(u)}{\wp^{*2}(u)} du = -\frac{\beta}{\lambda_-^2} \ln \left(\frac{2\pi - x}{\delta} \right) + \tilde{g}(x) \quad \text{for} \quad 2\pi - \delta \leq x \leq 2\pi \quad (5.13)$$

Since $x_1 = o(\delta)$, from (5.9), (5.13) and (5.12) we gather

$$\rho(x) = \rho(x_1) e^{\bar{g}(x)(1+g(x))} \left(\frac{x_1}{x} \right)^{\frac{\beta}{\lambda_+^2} (1+g(x))} \quad \text{for} \quad x_1 \leq x \leq \delta, \quad (5.14)$$

then, in particular, from (5.4),

$$\sup_{x_1 \leq x \leq \delta \wedge x_*} |\rho(x)| \leq |\rho(x_1)| = o(\eta) \quad (5.15)$$

thus $x_* > \delta$ then (5.5) follows from (5.14). From (5.5) we have, thus

$$\rho(x) = \rho(x_1) x_1^{\frac{\beta}{\lambda_+^2}} f^\delta(x) (1 + \mathcal{O}(\delta)) \quad \text{for } \delta \leq x \leq (2\pi - \delta) \wedge x_* \quad (5.16)$$

Since $f^\delta(x)$ does not depend on η , we have in particular that $\sup_{\delta \leq x \leq 2\pi - \delta} |\rho(x)| = o(\eta)$, hence $x_* > 2\pi - \delta$, thus (5.6) follows. Finally

$$\rho(x) = \rho(2\pi - \delta) e^{\tilde{g}(x)(1+g(x))} (2\pi - x)^{-\frac{\beta}{\lambda_-^2}(1+g(x))} \quad \text{for } 2\pi - \delta \leq x < 2\pi \quad (5.17)$$

thus

$$\begin{aligned} \sup_{2\pi - \delta \leq x \leq x'_1 \wedge x_*} |\rho(x)| &= |\rho(x'_1)| = |\rho(2\pi - \delta)| \mathcal{O}\left(\eta^{-\frac{\beta}{\lambda_-^2}}\right) \\ &= |\rho(x_1)| \mathcal{O}\left(\eta^{\beta\left(\frac{1}{\lambda_+^2} - \frac{1}{\lambda_-^2}\right)}\right) = o(\eta) \quad \text{as } \eta \rightarrow 0 \end{aligned} \quad (5.18)$$

since $\frac{\beta}{\lambda_+^2} > \frac{\beta}{\lambda_-^2}$. We have, thus, $x_* > x'_1$, then (5.7) follows from (5.16) and (5.17). \square

Remark 5.3. Notice that

$$\rho_k(x) = o(\eta) \quad \text{for any } x_k \leq x \leq x'_k \quad (5.19)$$

and, by (5.18),

$$\rho(x'_k) = |\rho(x_k)| \mathcal{O}\left(\eta^{\frac{2+\theta}{1+\theta}\theta}\right) \quad \text{as } \eta \rightarrow 0 \quad (5.20)$$

since $\frac{\beta}{\lambda_+^2} - \frac{\beta}{\lambda_-^2} = \frac{\gamma}{\beta} \sqrt{\gamma^2 + 4\beta} = \frac{2+\theta}{1+\theta} \theta$.

Lemma 5.4. For x_k, x'_k as in (5.3) and $\wp_k(x)$ satisfying (5.4),

$$\sup_{x_k \leq x \leq x'_k} \left| \frac{d}{dx} \wp_k(x) \right| = \mathcal{O}(1) \quad \text{as } \eta \rightarrow 0 \quad (5.21)$$

Proof. $\wp_k(x)$ verifies the equation

$$\frac{d}{dx} \wp_k(x) = -\frac{V'(x)}{\wp_k(x)} - \gamma \quad (5.22)$$

then it is sufficient to prove that $V'(x)/\wp_k(x)$ is uniformly bounded in η in a neighborhood of x_k and x'_k . Thus the result easily follows by expanding $V'(x)$ and $\wp_k(x)$ in a neighborhood of x_k and x'_k and by using Lemma 5.1 and Lemma 5.2. \square

By similar arguments can be proved the following Lemma.

Lemma 5.5. For x_k, x'_k as in (5.3) and $\wp_k(x)$ satisfying (5.4) we have

$$\left| \frac{d}{dx} \wp_k(x) - \frac{d}{dx} \wp_k^*(x) \right| \leq |\wp_k(x_k) - \wp_k^*(x_k)| \eta^{-1} \quad \text{for } x_k \leq x \leq x'_k \quad (5.23)$$

We define, now, the functionals

$$\Sigma_r^{(n)}[\wp, \bar{x}](x) := \int_{\bar{x}}^x \frac{du}{\wp(u)} e^{n \int_u^x \frac{V'(u')}{\wp^2(u')} du'} \quad (5.24)$$

and

$$\Sigma_y^{(n)}[\wp, \bar{x}](x) := \wp^n(x) \int_{\bar{x}}^x \frac{du}{\wp(u)} \left[\int_u^x \frac{du'}{\wp^2(u')} e^{\int_u^{u'} \frac{V'(u'')}{\wp^2(u'')} du''} \right]^n \quad (5.25)$$

$n \in \mathbb{N}$, $x \geq \bar{x}$. In the rest of the Section we provide some estimates on $\Sigma_r^{(n)}[\wp, \bar{x}](\cdot)$ and $\Sigma_y^{(n)}[\wp, \bar{x}](\cdot)$ that are fundamental for the study of the variances in Section 6.

For $\eta, x_k, x'_k, \wp_k(x)$ verifying the conditions in (5.3) and (5.4), we prove the following Lemma.

Lemma 5.6. *For any $k \geq 1, n \in \mathbb{N}$ we have*

$$\sup_{x_k \leq x \leq x'_k} \frac{\Sigma_r^{(n)}[\wp_k, x_k](x)}{\rho_k(x)^n} \rho_k(x_k)^n x_k^{n(1+\theta)} = \mathcal{O}(1) \quad \text{as} \quad \eta \rightarrow 0 \quad (5.26)$$

moreover

$$\Sigma_r^{(n)}[\wp_k, x_k](x'_k) \frac{\rho_k(x_k)^n x_k^{n(1+\theta)}}{\rho_k(x'_k)^n} = \Theta(1) \quad \text{as} \quad \eta \rightarrow 0 \quad (5.27)$$

Proof. As before, we prove the statement for $k = 1$ and omit the index 1 in the notation.

At first we prove that

$$\left| \frac{\Sigma_r^{(n)}[\wp, x_1](x)}{\rho(x)^n F_n[\wp, x_1](x)} - 1 \right| = o(\eta) \quad \text{with} \quad F_n[\wp, x_1](x) := \int_{x_1}^x \frac{du}{\wp(u) \rho(u)^n} \quad (5.28)$$

From (5.8) we have

$$\rho'(x) = \frac{V'(x)}{\wp(x)(\wp(x) - \rho(x))} \quad (5.29)$$

then

$$\rho(x) = \rho(u) e^{\int_u^x \frac{V'(u')}{\wp(u')(\wp(u') - \rho(u'))} du'} \quad \text{for any} \quad x_1 \leq u \leq x \quad (5.30)$$

thus, from (5.20) there exists $\hat{g}(x)$ such that

$$\rho(x) = \rho(u) e^{(1+\hat{g}(x)) \int_u^x \frac{V'(u')}{\wp^2(u')} du'} \quad \text{and} \quad \sup_{x_1 \leq x \leq x'_1} |\hat{g}(x)| = o(\eta) \quad (5.31)$$

it follows that

$$\sup_{x_1 \leq u \leq x \leq x'_1} \left| \frac{\rho(u)^n}{\rho(x)^n} e^{n \int_u^x \frac{V'(u')}{\wp(u')^2} du'} - 1 \right| = o(\eta) \quad (5.32)$$

this yields (5.28).

Let δ be small enough, $x \in [x_1, \delta]$, then by (5.1) and (5.5),

$$F_n[\wp, x_1](x) = (1 + \mathcal{O}(\delta)) \frac{x_1^{-\frac{n\beta}{\lambda_+^2}}}{\lambda_+ \rho(x_1)^n} \int_{x_1}^x u^{-1+\frac{n\beta}{\lambda_+^2}} du \quad (5.33)$$

$$= (1 + \mathcal{O}(\delta)) \frac{\lambda_+}{n\beta \rho(x_1)^n} \left[\left(\frac{x}{x_1} \right)^{\frac{n\beta}{\lambda_+^2}} - 1 \right] \quad \text{for} \quad x_1 \leq x \leq \delta \quad (5.34)$$

Let, now, $x \in [\delta, 2\pi - \delta]$, then $F_n[\wp, x_1](\delta) = F_n[\wp, x_1](\delta) + F_n[\wp, \delta](x)$ with, by (5.6) and (5.20),

$$F_n[\wp, \delta](x) = (1 + \mathcal{O}(\delta)) \frac{x_1^{-\frac{n\beta}{\lambda_+^2}}}{\rho(x_1)^n} \int_{\delta}^x \frac{du}{\wp^*(u) f^{\delta}(u)^n} \quad \text{for } \delta \leq x \leq 2\pi - \delta \quad (5.35)$$

with $f^{\delta}(x)$ as in Lemma 5.2. Finally, for $x \in [2\pi - \delta, x'_1]$, $F_n[\wp, x_1](x) = F_n[\wp, x_1](2\pi - \delta) + F_n[\wp, 2\pi - \delta](x)$, with, by (5.2) and (5.7),

$$F_n[\wp, 2\pi - \delta](x) = (1 + \mathcal{O}(\delta)) \frac{x_1^{-\frac{n\beta}{\lambda_+^2}}}{|\lambda_-| f^{\delta}(2\pi - \delta)^n \rho(x_1)^n} \int_{2\pi-x}^{\delta} u^{-1+\frac{n\beta}{\lambda_-^2}} du \quad (5.36)$$

$$= (1 + \mathcal{O}(\delta)) \frac{|\lambda_-| x_1^{-\frac{n\beta}{\lambda_+^2}}}{n\beta f^{\delta}(2\pi - \delta)^n \rho(x_1)^n} \left[\delta^{\frac{n\beta}{\lambda_-^2}} - (2\pi - x)^{\frac{n\beta}{\lambda_-^2}} \right] \quad \text{for } 2\pi - \delta \leq x \leq x'_1 \quad (5.37)$$

From (5.34), (5.35) and (5.37) we have that, for any $x \in [x_1, x'_1]$,

$$F_n[\wp, x_1](x) \rho(x_1)^n x_1^{\frac{n\beta}{\lambda_+^2}} = \mathcal{O}(1) \quad \text{as } \eta \rightarrow 0 \quad (5.38)$$

and, in particular,

$$\lim_{\eta \rightarrow 0} F_n[\wp, x_1](x'_1) \rho(x_1)^n x_1^{\frac{n\beta}{\lambda_+^2}} = (1 + \mathcal{O}(\delta)) \left(\frac{\lambda_+ \delta^{\frac{n\beta}{\lambda_+^2}}}{n\beta} + \int_{\delta}^{2\pi-\delta} \frac{du}{\wp^*(u) f^{\delta}(u)^n} + \frac{|\lambda_-| \delta^{\frac{n\beta}{\lambda_-^2}}}{n\beta f^{\delta}(2\pi - \delta)^n} \right) \quad (5.39)$$

Hence, by (5.38) and (5.28), for any $x \in [x_1, x'_1]$,

$$\frac{\Sigma_r^{(n)}[\wp, x_1](x)}{\rho(x)^n} \rho(x_1)^n x_1^{\frac{n\beta}{\lambda_+^2}} = \mathcal{O}(1) \quad \text{as } \eta \rightarrow 0 \quad (5.40)$$

then (5.26) follows since $\beta/\lambda_+^2 = 1 + \theta$. From (5.39) and the definition of $f^{\delta}(x)$ it is clear that the limit

$$\lim_{\eta \rightarrow 0} F_n[\wp, x_1](x'_1) \rho(x_1)^n x_1^{\frac{n\beta}{\lambda_+^2}} = \quad (5.41)$$

$$\lim_{\delta \rightarrow 0} \left(\delta^{\frac{n\beta}{\lambda_+^2}} \int_{\delta}^{2\pi-\delta} \frac{du}{\wp^*(u)} e^{-n \int_{\delta}^u \frac{V'(u')}{\wp^{*2}(u')} du'} + \frac{|\lambda_-| \delta^{n\beta(\lambda_+^{-2} + \lambda_-^{-2})}}{n\beta} e^{-n \int_{\delta}^{2\pi-\delta} \frac{V'(u')}{\wp^{*2}(u')} du'} \right) \quad (5.42)$$

must be finite and strictly positive, thus (5.27) follows from (5.41) and (5.28). \square

Corollary 5.7. *From (5.26) and (5.3) it follows that*

$$\sup_{x_k \leq x \leq x'_k} \Sigma_r^{(n)}[\wp_k, x_k](x) = \mathcal{O}(\eta^{-\frac{n}{1+\theta}}) \quad \text{as } \eta \rightarrow 0 \quad (5.43)$$

and

$$\Sigma_r^{(n)}[\wp_k, x_k](x'_k) = \Theta(\eta^{-\frac{n}{1+\theta}}) \quad \text{as } \eta \rightarrow 0 \quad (5.44)$$

Proof. The result follows since, from Lemma 5.2,

$$\sup_{x_k \leq x \leq x'_k} \frac{\rho_k(x)}{\rho_k(x_k)} = \frac{\rho_k(x'_k)}{\rho_k(x_k)} \quad (5.45)$$

where

$$\frac{\rho_k(x'_k)}{\rho_k(x_k)} = \Theta \left(\eta^{\frac{2+\theta}{1+\theta} \theta} \right) \quad (5.46)$$

since $\frac{\beta}{\lambda_+^2} - \frac{\beta}{\lambda_-^2} = \frac{\gamma}{\beta} \sqrt{\gamma^2 + 4\beta} = \frac{2+\theta}{1+\theta} \theta$. \square

Lemma 5.8. *We have*

$$\sup_{x_k \leq x \leq x'_k} |\Sigma_y^{(n)}[\wp_k, x_k](x)| = \mathcal{O}(\eta^{-n}) \quad \text{as} \quad \eta \rightarrow 0 \quad (5.47)$$

Proof. We prove the result for $k = 1$ and omit the index 1. From (5.32), it follows that

$$\sup_{x_1 \leq x \leq x'_1} \left| \frac{\Sigma_y^{(n)}[\wp, x_1]}{\wp(x)^n} \left\{ \int_{x_1}^x \frac{B(u, x)^n du}{\wp(u)\rho(u)^n} \right\}^{-1} - 1 \right| = o(\eta) \quad \text{as} \quad \eta \rightarrow 0 \quad (5.48)$$

with

$$B(u, x) := \int_u^x \frac{\rho(u') du'}{\wp(u')^2} \quad (5.49)$$

By the same techniques used in the proof of the previous Lemma it is possible to show that there exists $c > 0$ such that

$$B(u, x) \leq c \rho(x_1) \eta^{\frac{\beta}{\lambda_+^2}} \left[u^{-(1+\frac{\beta}{\lambda_+^2})} \mathbf{1}_{x_1 \leq u \leq \delta} + \mathbf{1}_{\delta \leq u \leq 2\pi-\delta} + (2\pi-u)^{-(1+\frac{\beta}{\lambda_-^2})} \mathbf{1}_{2\pi-\delta \leq u \leq x'_1} \right] \quad (5.50)$$

for any η small enough, then, by Lemma 5.2, there exists $c > 0$ such that

$$\frac{B(u, x)}{\rho(u)} \leq c \left[u^{-1} \mathbf{1}_{x_1 \leq u \leq \delta} + \mathbf{1}_{\delta \leq u \leq 2\pi-\delta} + (2\pi-u)^{-1} \mathbf{1}_{2\pi-\delta \leq u \leq x'_1} \right] \quad (5.51)$$

hence

$$\sup_{x_1 \leq x \leq x'_1} \int_{x_1}^x \frac{B(u, x)^n du}{\wp(u)\rho(u)^n} = \mathcal{O}(\eta^{-n}) \quad \text{as} \quad \eta \rightarrow 0 \quad (5.52)$$

thus (5.47) follows from (5.48) and (5.52). \square

6 Estimates of the variances

We recall that $(X_k(t), P_k(t))$, $T_k \leq t \leq S_k$ is the solution of the problem

$$\begin{cases} \dot{X}_k = P_k & X_k(T_k) = x_k \\ \dot{P}_k = -\gamma P_k - V'(X_k) & P_k(T_k) = p_k \end{cases} \quad (6.1)$$

$\wp_k(x)$ is the related orbit in $[2(k-1)\pi, 2k\pi]$, i.e. the path such that $\wp_k(X_k(t)) = P_k(t)$ for $T_k \leq t \leq S_k$ and $\omega_k(t) = \frac{d}{dx} \wp_k(X_k(t))$.

In this Section we provide some estimates on the variances of the processes $\bar{y}_k(t)$ and $\bar{r}_k(t)$. The two following Lemmas follow directly from their definitions in (3.15) and (3.14).

Lemma 6.1. *Let*

$$\sigma_r^2(T_k, t) := \epsilon^2 \int_{T_k}^t e^{-2 \int_s^t (\omega_k(s') + \gamma) ds'} ds \quad (6.2)$$

then

$$\bar{r}_k(t) | T_k \sim \mathbf{Gauss}(0, \sigma_r(T_k, t)), \quad t \geq T_k$$

Lemma 6.2. *Let*

$$\sigma_y^2(t, T_k) := \epsilon^2 \int_{T_k}^t e^{2 \int_s^t \omega_k(s'') ds''} \left[\int_s^t e^{-\int_s^{s'} (2\omega_k(s'') + \gamma) ds''} ds' \right]^2 ds \quad (6.3)$$

then

$$\bar{y}_k(t) | T_k \sim \mathbf{Gauss}(0, \sigma_y(T_k, t)), \quad t \geq T_k$$

Let $\eta_\epsilon > 0$ be as in Section 3, we recall that

$$\mathcal{H}_k^\xi = \{(x, p) : p = \eta_\epsilon + \lambda_-(x - 2(k-1)\pi), |p - \lambda_+(x - 2(k-1)\pi)| \leq \bar{\sigma}_\epsilon \epsilon^{-\xi}\}, \quad \text{for } k \geq 1 \quad (6.4)$$

and $\mathcal{H}_0 = \{(-\pi, \wp_0^*(-\pi))\}$, and define the stopping time

$$\bar{S}_k := \inf \left\{ t \geq T_k : X_k(t) \geq 2k\pi - \frac{\eta_\epsilon}{2(\lambda_+ - \lambda_-)} \right\} \quad (6.5)$$

We denote by $x'_k := X(\bar{S}_k)$, then

$$2k\pi - x'_k = \frac{\eta_\epsilon}{2(\lambda_+ - \lambda_-)} \quad (6.6)$$

and $x_k \leq X_k(t) \leq x'_k$ for $T_k \leq t \leq \bar{S}_k$.

In the following Lemma we prove that, if $(x_k, p_k) \in \mathcal{H}_k^\xi$, then x_k , x'_k and $\wp_k(\cdot)$ satisfy the conditions (5.3) and (5.4) in $[2(k-1)\pi, 2k\pi]$ in the following sense.

Lemma 6.3. *Let $(x_k, p_k) \in \mathcal{H}_k^\xi$, with ξ small enough, then*

$$x_k - 2(k-1)\pi = \Theta(\eta_\epsilon) \quad \text{and} \quad 2k\pi - x'_k = \Theta(\eta_\epsilon) \quad \text{as} \quad \epsilon \rightarrow 0 \quad (6.7)$$

moreover

$$|\wp_k(x_k) - \wp_k^*(x_k)| = o(\eta_\epsilon) \quad \text{as} \quad \epsilon \rightarrow 0 \quad (6.8)$$

Proof. (6.7) follows directly from the definition of \mathcal{H}_k^ξ and from (6.6), whereas (6.8) is verified since

$$\begin{aligned} & |\wp_k(x_k) - \wp_k^*(x_k)| = |p_k - \wp_k^*(x_k)| \\ & \leq |p_k - \lambda_+(x_k - 2(k-1)\pi)| + |\lambda_+(x_k - 2(k-1)\pi) - \wp_k^*(x_k)| \\ & \leq \bar{\sigma}_\epsilon \epsilon^{-\xi} + c\eta_\epsilon^2 \leq 2\bar{\sigma}_\epsilon \epsilon^{-\xi} \end{aligned} \quad (6.9)$$

where the last inequality follows from the definition of \mathcal{H}_k^ξ , (5.1) and the left hand side of (6.7). Hence (6.8) holds if ξ is small enough, since, from (3.22), $\bar{\sigma}_\epsilon = o(\eta_\epsilon)$ as $\epsilon \rightarrow 0$. \square

Lemma 6.4. *Let $(x_k, p_k) \in \mathcal{H}_k^\xi$, with ξ small enough, then*

$$\wp_k(x'_k) = \lambda_-(x'_k - 2k\pi) + \mathcal{O}(\bar{\sigma}_\epsilon \epsilon^{-\xi}) \quad \text{as} \quad \epsilon \rightarrow 0 \quad (6.10)$$

Proof. From (5.20) and (6.9) we have

$$|\wp_k(x'_k) - \wp_k^*(x'_k)| \leq c \eta_\epsilon^{\frac{2+\theta}{1+\theta}} \bar{\sigma}_\epsilon \epsilon^{-\xi} = \mathcal{O}(\tilde{\sigma}_\epsilon \epsilon^{-\xi}) \quad \text{as } \epsilon \rightarrow 0 \quad (6.11)$$

then the result follows from (5.2) since, from (3.22), $\eta_\epsilon^2 = o(\tilde{\sigma}_\epsilon)$ as $\epsilon \rightarrow 0$. \square

Remark 6.5. As a consequence of Lemma 6.3 and the considerations done in Section 4.1, $\wp_k(\cdot)$ is well defined in $[x_k, x'_k]$.

Proposition 6.6. Let $(x_k, p_k) \in \mathcal{H}_k^\xi$, ξ small enough, then

$$\sup_{T_k \leq t \leq \bar{S}_k} \sigma_r^2(T_k, t) = \mathcal{O}(\sigma_\epsilon^2) \quad \text{as } \epsilon \rightarrow 0 \quad \mathbf{P}_{T_k, x_k, p_k} \text{ a.s.} \quad (6.12)$$

Proof. We have

$$\sigma_r^2(T_k, t) = \epsilon^2 \int_{x_k}^{X_k(t)} e^{-2 \int_u^{X_k(t)} \frac{\wp'_k(u') + \gamma}{\wp_k(u')} du'} \frac{du}{\wp_k(u)} = \epsilon^2 \Sigma_r^{(2)}[\wp_k, x_k](X_k(t)) \quad (6.13)$$

where the second identity follows from (2.5), then

$$\sup_{T_k \leq t \leq \bar{S}_k} \sigma_r^2(T_k, t) = \epsilon^2 \sup_{x_k \leq x \leq x'_k} \Sigma_r^{(2)}[\wp_k, x_k](x) = \epsilon^2 \mathcal{O}(\eta_\epsilon^{-\frac{2}{1+\theta}}) \quad (6.14)$$

where the last equivalence follows from (5.43). Then (6.12) follows from the definition of σ_ϵ in (3.19). \square

Proposition 6.7. Let $(x_k, p_k) \in \mathcal{H}_k^\xi$, ξ small enough, then

$$\sup_{T_k \leq t \leq \bar{S}_k} \sigma_y^2(T_k, t) = \mathcal{O}(\epsilon^2 \eta_\epsilon^{-2}) \quad \mathbf{P}_{T_k, x_k, p_k} \text{ a.s.} \quad (6.15)$$

Proof. We have

$$\sigma_y^2(T_k, t) = \epsilon^2 \int_{T_k}^t e^{2 \int_{X_k(s)}^{X_k(t)} \frac{\wp'_k(u')}{\wp_k(u')} du'} \left[\int_s^t e^{-\int_{X_k(s')}^{X_k(s)} \frac{2\wp'_k(u') + \gamma}{\wp_k(u')} du'} ds' \right]^2 ds \quad (6.16)$$

$$= \wp_k^2(X_k(t)) \int_{x_k}^{X_k(t)} \wp_k(u) \left[\int_u^{X_k(t)} \frac{du'}{\wp_k^3(u')} e^{-\int_u^{u'} \frac{\gamma}{\wp_k(u'')} du''} \right]^2 du \quad (6.17)$$

$$= \epsilon^2 \Sigma_y^{(2)}[\wp_k, x_k](X_k(t)) \quad (6.18)$$

thus

$$\sup_{T_k \leq t \leq \bar{S}_k} \sigma_y^2(T_k, t) = \epsilon^2 \sup_{x_k \leq x \leq x'_k} \Sigma_y^{(2)}[\wp_k, x_k](x) = \epsilon^2 \mathcal{O}(\eta_\epsilon^{-2}) \quad (6.19)$$

where the last equivalence follows from (5.47). \square

Lemma 6.8. *Let $(x_k, p_k) \in \mathcal{H}_k^\xi$, ξ small enough, then there exists $C > 0$ such that*

$$\sup_{\epsilon < 1} \sup_{T_k \leq t \leq \bar{S}_k} |\omega_k(t)| \leq C \quad (6.20)$$

Proof. It is a direct consequence of Lemma 5.4, since

$$\sup_{T_k \leq t \leq \bar{S}_k} |\omega_k(t)| = \sup_{x_k \leq x \leq x'_k} \left| \frac{d}{dx} \wp_k(x) \right| \quad (6.21)$$

□

Lemma 6.9. *Let*

$$H_r(T_k, t) := \int_{T_k}^t e^{-\int_s^t (\omega_k(s') + \gamma) ds'} ds \quad (6.22)$$

and

$$H_y(t, T_k) := \int_{T_k}^t e^{\int_s^t \omega_k(s'') ds''} \left[\int_s^t e^{-\int_s^{s'} (2\omega_k(s'') + \gamma) ds''} ds' \right] ds \quad (6.23)$$

then, for $(x_k, p_k) \in \mathcal{H}_k^\xi$, ξ small enough, we have

$$\sup_{T_k \leq t \leq \bar{S}_k} H_r(T_k, t) = \mathcal{O} \left(\eta_\epsilon^{-\frac{1}{1+\theta}} \right) \quad (6.24)$$

and

$$\sup_{T_k \leq t \leq \bar{S}_k} H_y(T_k, t) = \mathcal{O} (\eta_\epsilon^{-1}) \quad (6.25)$$

Proof. It follows directly from (5.43) and (5.47) since

$$H_r(T_k, t) = \Sigma_r^{(1)}[\wp_k, x_k](X_k(t)) \quad \text{and} \quad H_y(T_k, t) = \Sigma_y^{(1)}[\wp_k, x_k](X_k(t)) \quad (6.26)$$

□

7 Estimate of Errors

In this Section we prove Propositions 4.6, 4.7, 4.8, 4.9 and 4.10.

Errors in the stable interval

We denote by $\mathbf{P}_{T_k, x_k, p_k}$ the law of $(x(t), p(t))$ given $x(T_k) = x_k$, $p(T_k) = p_k$. In this first part of the Section we will prove the following Proposition.

Proposition 7.1. *There exists $C > 0$ such that, for any $\zeta > 0$, ξ, ϵ small enough,*

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ \sup_{T_k \leq t \leq S_k} |y_k(t)| \geq \frac{\epsilon}{\eta_\epsilon} \epsilon^{-\zeta} \right\} \leq e^{-C\epsilon^{-2\zeta}} \quad (7.1)$$

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ \sup_{T_k \leq t \leq S_k} |\mathcal{R}_k(t)| \geq \frac{\epsilon \sigma_\epsilon}{\eta_\epsilon^2} \epsilon^{-2\zeta} \right\} \leq e^{-C\epsilon^{-2\zeta}} \quad (7.2)$$

and

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ \sup_{T_k \leq t \leq S_k} |r_k(t)| \geq \sigma_\epsilon \epsilon^{-\zeta} \right\} \leq e^{-C\epsilon^{-2\zeta}} \quad (7.3)$$

Lemma 7.2. *There exists $C > 0$ such that, for any $\zeta > 0$, ξ, ϵ small enough,*

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ \sup_{T_k \leq t \leq \bar{S}_k} |y_k(t)| \geq \frac{\epsilon}{\eta_\epsilon} \epsilon^{-\zeta} \right\} \leq e^{-C\epsilon^{-2\zeta}} \quad (7.4)$$

Proof. Suppose that $(x_k, p_k) \in \mathcal{H}_k^\xi$ for some $\xi > 0$ small enough. Let us fix $\zeta > 0$, then, from Lemma 6.2, (8.4) and (6.15) it follows that there exists $C > 0$ such that, for any ϵ small enough,

$$\mathbf{P}_{T_k, x_k, p_k} \left\{ \sup_{T_k \leq t \leq \bar{S}_k} |\bar{y}_k(t)| \geq \frac{\epsilon}{2\eta_\epsilon} \epsilon^{-\zeta} \right\} \leq e^{-C\epsilon^{-2\zeta}} \quad (7.5)$$

We recall that $y_k(t) = \bar{y}_k(t) + \mathcal{Y}_k(t)$ with

$$\mathcal{Y}_k(t) \leq H_y(T_k, t) \cdot \sup_{T_k \leq s \leq \bar{S}_k} \varphi(X_k(s), y_k(s)) \quad \text{for } T_k \leq t \leq \bar{S}_k \quad (7.6)$$

(see (3.16) and (6.25)). From (3.7), we know that $\varphi(x, y) = \mathcal{O}(y^2)$ for small $|y|$, then if we define the stopping time $\Upsilon_k^\zeta := \inf\{t \geq T_k : |y_k(t)| \geq \epsilon^{1-\zeta} \eta_\epsilon^{-1}\}$, then there exists $c > 0$ such that

$$\sup_{T_k \leq s \leq \Upsilon_k^\zeta \wedge \bar{S}_k} |\varphi(X_k(s), y_k(s))| \leq c \frac{\epsilon^{2(1-\zeta)}}{\eta_\epsilon^2} \quad (7.7)$$

thus, from (7.6) and (6.25) we have

$$|\mathcal{Y}_k(t)| \leq c \frac{\epsilon^{2(1-\zeta)}}{\eta_\epsilon^3} \quad \text{for any } T_k \leq t \leq \Upsilon_k^\zeta \wedge \bar{S}_k. \quad (7.8)$$

Thus, since $|y_k(t)| \leq |\bar{y}_k(t)| + |\mathcal{Y}_k(t)|$, by (7.5) and (7.8), with $\mathbf{P}_{T_k, x_k, p_k}$ -probability greater than $1 - 2e^{-C\epsilon^{-2\zeta}}$,

$$\sup_{T_k \leq t \leq \Upsilon_k^\zeta \wedge \bar{S}_k} |y_k(t)| \leq \frac{\epsilon^{1-\zeta}}{2\eta_\epsilon} + c \frac{\epsilon^{2(1-\zeta)}}{\eta_\epsilon^3} < \frac{\epsilon^{1-\zeta}}{\eta_\epsilon},$$

hence, in particular, with the same probability, $\bar{S}_k < \Upsilon_k^\zeta$, then (7.4) follows. \square

Lemma 7.3. *There exists $C > 0$ such that, for any $\zeta > 0$, ξ, ϵ small enough,*

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ \sup_{T_k \leq t \leq \bar{S}_k} |\mathcal{R}_k(t)| \geq \frac{\epsilon \sigma_\epsilon}{\eta_\epsilon^2} \epsilon^{-2\zeta} \right\} \leq e^{-C\epsilon^{-2\zeta}} \quad (7.9)$$

Proof. Suppose that $(x_k, p_k) \in \mathcal{H}_k^\xi$ for some ξ small enough. From (3.17) we know that

$$\mathcal{R}_k(t) = H_r(T_k, t) \cdot \sup_{T_k \leq s \leq \bar{S}_k} \varphi(X_k(s), y_k(s)) ds \quad (7.10)$$

thus, by Lemma 7.2, we know that there exists $c > 0$ such that

$$\sup_{T_k \leq s \leq \bar{S}_k} |\varphi(X_k(s), y_k(s))| \leq c \frac{\epsilon^{2(1-\zeta)}}{\eta_\epsilon^2} \quad (7.11)$$

with $\mathbf{P}_{T_k, x_k, p_k}$ -probability greater than $1 - e^{-C\epsilon^{-2\zeta}}$, then, from (7.10), (7.11) and (6.24) it follows that

$$\sup_{T_k \leq t \leq \bar{S}_k} |\mathcal{R}_k(t)| \leq c \frac{\epsilon^{2(1-\zeta)}}{\eta_\epsilon^2} \eta_\epsilon^{-\frac{1}{1+\theta}} \quad (7.12)$$

with $\mathbf{P}_{T_k, x_k, p_k}$ -probability greater than $1 - e^{-C\epsilon^{-2\zeta}}$, hence (7.9) follows from the definition of σ_ϵ . \square

Lemma 7.4. *There exists $C > 0$ such that, for any $\zeta > 0$, ξ, ϵ small enough,*

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ \sup_{T_k \leq t \leq \bar{S}_k} |r_k(t)| \geq \sigma_\epsilon \epsilon^{-\zeta} \right\} \leq e^{-C\epsilon^{-2\zeta}} \quad (7.13)$$

Proof. Assume $(x_k, p_k) \in \mathcal{H}_k^\xi$, with ξ small enough. We recall that $r_k(t) = \bar{r}_k(t) + \mathcal{R}_k(t)$. From Lemma 6.1, (8.4) and (6.12), there exists $C > 0$ such that, for any $\zeta > 0$, ϵ small enough,

$$\mathbf{P}_{T_k, x_k, p_k} \left\{ \sup_{T_k \leq t \leq \bar{S}_k} |\bar{r}_k(t)| \geq \frac{\sigma_\epsilon}{2} \epsilon^{-\zeta} \right\} \leq e^{-C\epsilon^{-2\zeta}} \quad (7.14)$$

hence the result descends from (7.14) and Lemma 7.3, since, from (3.22), $\epsilon = o(\eta_\epsilon^2)$. \square

Corollary 7.5. *There exists $C > 0$ such that, for any $\zeta > 0$, ξ, ϵ small enough,*

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ \sup_{T_k \leq t \leq \bar{S}_k} |q_k(t)| \geq \frac{\epsilon}{\eta_\epsilon} \epsilon^{-\zeta} \right\} \leq e^{-C\epsilon^{-2\zeta}} \quad (7.15)$$

Proof. We recall that $q_k(t) = r_k(t) + \omega_k(t)y_k(t)$, then (7.15) easily follows from Lemma 7.2, Lemma 7.4 and Lemma 6.8. \square

Lemma 7.6. *Let \tilde{S}_k be the stopping time defined in (6.5) and*

$$\tilde{S}_k := \inf \left\{ t \geq T_k : X_k(t) \geq 2k\pi - \frac{2\eta_\epsilon}{(\lambda_+ - \lambda_-)} \right\} \quad (7.16)$$

then there exists $C > 0$ such that, for any $\zeta > 0$, ξ, ϵ small enough,

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ \tilde{S}_k \leq S_k \leq \bar{S}_k \right\} \geq 1 - e^{-C\epsilon^{-2\zeta}} \quad (7.17)$$

Proof. We only show that

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ S_k \leq \bar{S}_k \right\} \geq 1 - e^{-C\epsilon^{-2\zeta}} \quad (7.18)$$

since the arguments for the estimate of $\mathbf{P}_{T_k, x_k, p_k} \left\{ \tilde{S}_k \leq S_k \right\}$ are specular. Assume $(x_k, p_k) \in \mathcal{H}_k^\xi$, for some ξ small enough. We have

$$\mathbf{P}_{T_k, x_k, p_k} \left\{ S_k \leq \bar{S}_k \right\} \geq \mathbf{P}_{T_k, x_k, p_k} \left\{ v_k(\bar{S}_k) \leq \eta_\epsilon \right\} \quad (7.19)$$

where

$$v_k(t) = P_k(t) - \lambda_+(X_k(t) - 2k\pi) + q_k(t) - \lambda_+ y_k(t). \quad (7.20)$$

Recalling the definition of \bar{S}_k and x'_k in (6.5), we have

$$v_k(\bar{S}_k) = \wp_k(x'_k) - \lambda_+(x'_k - 2k\pi) + q_k(\bar{S}_k) - \lambda_+ y_k(\bar{S}_k). \quad (7.21)$$

thus, from Lemma 6.4 we have

$$\left| v_k(\bar{S}_k) - \frac{\eta_\epsilon}{2} - (q_k(\bar{S}_k) - \lambda_+ y_k(\bar{S}_k)) \right| \leq c \tilde{\sigma}_\epsilon \epsilon^{-\xi} \quad (7.22)$$

for some $c > 0$. By Lemma 7.2 and Corollary 7.5, there exists $C > 0$ such that, for any ζ, ϵ small enough,

$$\mathbf{P}_{T_k, x_k, p_k} \left\{ |q_k(\bar{S}_k) - \lambda_+ y_k(\bar{S}_k)| \geq \frac{\epsilon}{\eta_\epsilon} \epsilon^{-\zeta} \right\} \leq \epsilon^{-C\epsilon^{-2\zeta}} \quad (7.23)$$

From (3.22) we know that $\tilde{\sigma}_\epsilon \epsilon^{-\xi} = o(\eta_\epsilon)$ and $\epsilon^{1-\xi} = o(\eta_\epsilon^2)$, for ξ small enough, thus, from (7.22) and (7.23) it follows that

$$\mathbf{P}_{T_k, x_k, p_k} \{ v_k(\bar{S}_k) > \eta_\epsilon \} \leq \epsilon^{-C\epsilon^{-2\zeta}} \quad (7.24)$$

then (7.18) follows from (7.19) and (7.24). \square

Proof of Proposition 7.1. It directly follows from (7.4), (7.9), (7.13) and Lemma 7.6. \square

Errors in the critical interval

For any $\xi > 0$ we recall that

$$\mathcal{K}_k^\xi = \{ (x, p) : p - \lambda_+(x - 2k\pi) = \eta_\epsilon, |p - \lambda_-(x - 2k\pi)| \leq \sigma_\epsilon \epsilon^{-\xi} \}. \quad (7.25)$$

and denote by $\mathbf{P}_{S_k, \bar{x}_k, \bar{p}_k}$ the law of $(x(t), p(t))$ given $x(S_k) = \bar{x}_k$, $p(S_k) = \bar{p}_k$. We will prove the following result.

Proposition 7.7. *We have*

$$\sup_{S_k \leq t \leq T_{k+1}} |z_k(t)| \leq \eta_\epsilon \quad \mathbf{P}_{S_k, \bar{x}_k, \bar{p}_k} \text{ a.s.} \quad (7.26)$$

moreover there exists $C > 0$ such that, for any $\delta > 0$, ζ, ϵ small enough,

$$\mathbf{P}_{S_k, \bar{x}_k, \bar{p}_k} \left\{ \sup_{S_k \leq t \leq T_{k+1}} |v_k(t)| \geq (1 + \delta)\eta_\epsilon \right\} \leq e^{-C\epsilon^{-2\zeta}} \quad (7.27)$$

for any $(\bar{x}_k, \bar{p}_k) \in \mathbb{R}^2$.

Lemma 7.8. *There exists $C > 0$ such that, for any $\delta, \zeta > 0$, ϵ small enough,*

$$\mathbf{P}_{S_k, \bar{x}_k, \bar{p}_k} \left\{ \sup_{t \geq S_k} |\bar{v}_k(t)| \leq (1 + \delta)\eta_\epsilon \right\} \geq 1 - e^{-C\epsilon^{-2\zeta}} \quad (7.28)$$

for any $(\bar{x}_k, \bar{p}_k) \in \mathbb{R}^2$.

Proof. From Lemma 4.1 we know that the probability law of $\bar{v}_k(S_k+t) - \eta_\epsilon e^{\lambda_- t}$ is a centered Gaussian of variance $\sigma_v^2(t)$, independently on the initial condition (\bar{x}_k, \bar{p}_k) at time S_k . By (4.1) we have

$$\sup_{t \geq 0} \sigma_v^2(t) \leq \frac{\epsilon^2}{2|\lambda_-|} \quad (7.29)$$

thus, by (8.4), there exists $C > 0$ such that,

$$\mathbf{P}_{S_k, \bar{x}_k, \bar{p}_k} \left\{ \sup_{t \geq 0} |\bar{v}_k(S_k+t) - \eta_\epsilon e^{\lambda_- t}| \geq \epsilon^{1-\zeta} \right\} \leq e^{-C\epsilon^{-2\zeta}} \quad (7.30)$$

for any $\zeta, \delta > 0$, ϵ small enough, $(\bar{x}_k, \bar{p}_k) \in \mathbb{R}^2$, then (7.28) follows since, from (3.22), $\epsilon^{1-\zeta} = o(\eta_\epsilon)$. \square

Lemma 7.9. *There exists $C > 0$ such that, for any $\xi > 0$, ζ, ϵ small enough,*

$$\mathbf{1}_{(\bar{x}_k, \bar{p}_k) \in \mathcal{K}_k^\xi} \mathbf{P}_{S_k, \bar{x}_k, \bar{p}_k} \left\{ \sup_{t \geq S_k} |\bar{z}_k(t)| e^{-\lambda_+(t-S_k)} \geq \sigma_\epsilon \epsilon^{-\xi} \right\} \leq e^{-C\epsilon^{-2\zeta}} \quad (7.31)$$

Proof. From Lemma 4.2 we know that the probability law of the process $\bar{z}_k(S_k+t) - z_k(S_k)e^{\lambda_+ t}$ is a centered Gaussian of variance $\sigma_z^2(t)$ independent of the values (\bar{x}_k, \bar{p}_k) at time S_k . From the formula in (4.2) we have

$$\sup_{t \geq 0} \sigma_z^2(t) e^{-2\lambda_+ t} \leq \frac{\epsilon^2}{2\lambda_+} \quad (7.32)$$

thus, by (8.4), there exists $C > 0$ such that, for any $\zeta > 0$, ϵ small enough, $(\bar{x}_k, \bar{p}_k) \in \mathbb{R}^2$,

$$\mathbf{P}_{S_k, \bar{x}_k, \bar{p}_k} \left\{ \sup_{t \geq 0} |\bar{z}_k(S_k+t) e^{-\lambda_+ t} - z_k(S_k)| \geq \epsilon^{1-\zeta} \right\} \leq e^{-C\epsilon^{-2\zeta}} \quad (7.33)$$

then, with $\mathbf{P}_{S_k, \bar{x}_k, \bar{p}_k}$ -probability larger than $1 - e^{-C\epsilon^{-2\zeta}}$

$$|\bar{z}_k(t)| e^{-\lambda_+(t-S_k)} \leq |z_k(S_k)| + \epsilon^{1-\zeta} \quad \forall t \geq S_k. \quad (7.34)$$

Let us assume $(\bar{x}_k, \bar{p}_k) \in \mathcal{K}_k^\xi$, for some $\xi > 0$, hence (7.31) follows since $|z_k(S_k)| = |p - \lambda_-(x - 2k\pi)| \leq \sigma_\epsilon \epsilon^{-\xi}$ and since, from (3.22), $\epsilon^{1-\zeta} = o(\sigma_\epsilon \epsilon^{-\xi})$, for ζ small enough. \square

Proof of Propositions 7.7, 4.8 and 4.9. (7.26) follows directly from the definitions of T_{k+1} and \mathcal{K}_k^ξ , since $\sigma_\epsilon \epsilon^{-\xi} = o(\eta_\epsilon)$ for ξ small enough.

Assume $(\bar{x}_k, \bar{p}_k) \in \mathcal{K}_k^\xi$, thus $|x - 2k\pi| < 3\eta_\epsilon/(\lambda_+ - \lambda_-)$. Consider the stopping time

$$\Gamma_k := \inf \left\{ t \geq S_k : |x(t) - 2k\pi| \geq \frac{3\eta_\epsilon}{\lambda_+ - \lambda_-} \right\},$$

then, since $\psi_k(x) = \mathcal{O}((x - 2k\pi)^2)$ for small $(x - 2k\pi)$, by (3.4) there exists $c > 0$ such that, for any $S_k \leq t \leq \Gamma_k \wedge T_{k+1}$,

$$|\mathcal{V}_k(t)| \leq c\eta_\epsilon^2 e^{\lambda_- t} \int_{S_k}^t e^{-\lambda_- s} ds \leq c\eta_\epsilon^2 \quad (7.35)$$

and

$$|\mathcal{Z}_k(t)| \leq c\eta_\epsilon^2 e^{\lambda_+ t} \int_{S_k}^t e^{-\lambda_+ s} ds \leq c\eta_\epsilon^2 e^{\lambda_+(t-S_k)} \quad (7.36)$$

then

$$\sup_{S_k \leq t \leq \Gamma_k \wedge T_{k+1}} |\mathcal{V}_k(t)| \leq c'\eta_\epsilon^2 \quad \text{and} \quad \sup_{S_k \leq t \leq \Gamma_k \wedge T_{k+1}} |\mathcal{Z}_k(t)| e^{-\lambda_+(t-S_k)} \leq c'\eta_\epsilon^2 \quad (7.37)$$

for suitable $c' > 0$. Recalling that $z_k(t) = \bar{z}_k(t) + \mathcal{Z}_k(t)$, since $\eta_\epsilon^2 = o(\sigma_\epsilon)$, it follows from (7.31) and the right hand side of (7.37) that, for any $\xi > 0$, ζ, ϵ small enough,

$$\mathbf{1}_{(\bar{x}_k, \bar{p}_k) \in \mathcal{K}_k^\xi} \mathbf{P}_{S_k, \bar{x}_k, \bar{p}_k} \left\{ \sup_{S_k \leq t \leq \Gamma_k \wedge T_{k+1}} |z_k(t)| e^{-\lambda_+(t-S_k)} \geq \sigma_\epsilon \epsilon^{-\xi} \right\} \leq e^{-C\epsilon^{-2\zeta}} \quad (7.38)$$

for a suitable $C > 0$.

On the other hand we have $v_k(t) = \bar{v}_k(t) + \mathcal{V}_k(t)$, then, by Lemma 7.8 and (7.37), there exists $C > 0$ such that, for any $(\bar{x}_k, \bar{p}_k) \in \mathbb{R}^2$, $\zeta, \delta > 0$, ϵ small enough,

$$\mathbf{P}_{S_k, \bar{x}_k, \bar{p}_k} \left\{ \sup_{S_k \leq t \leq \Gamma_k \wedge T_{k+1}} |v_k(t)| \geq \eta_\epsilon(1 + \delta) \right\} \leq e^{-C\epsilon^{-2\zeta}} \quad (7.39)$$

We recall that $x(t) - 2k\pi = (z_k(t) - v_k(t))/(\lambda_+ - \lambda_-)$, thus from (7.26) and (7.39) it follows that

$$\mathbf{P}_{S_k, \bar{x}_k, \bar{p}_k} \{ \Gamma_k > T_{k+1} \} \geq \mathbf{P}_{S_k, \bar{x}_k, \bar{p}_k} \left\{ \sup_{S_k \leq t \leq \Gamma_k \wedge T_{k+1}} |x(t) - 2k\pi| \leq \frac{(2 + \delta)\eta_\epsilon}{\lambda_+ - \lambda_-} \right\} \geq 1 - e^{-C\epsilon^{-2\zeta}} \quad (7.40)$$

thus (7.27) follows from (7.39) and (7.40), whereas (4.18) follows from (7.38) and (7.40). (4.16) and (4.17) descend both from (7.35) and (7.40). \square

Conclusion of the Proofs

We conclude the proofs of Propositions 4.7, 4.6 and 4.10.

Proposition 7.10. *Suppose that $(x_k, p_k) \in \mathcal{H}_k^\xi$ with ξ small enough, then there exist $c' > c > 0$ such that*

$$c \frac{\eta_\epsilon^{(1+\theta)}}{\epsilon} \leq \frac{|P_k(S_k) - \wp_k^*(X_k(S_k))|}{\sigma_r(T_k, S_k) |\wp_k(\bar{x}) - \wp_k^*(\bar{x})|} \leq c' \frac{\eta_\epsilon^{(1+\theta)}}{\epsilon} \quad (7.41)$$

for any ϵ small enough.

Proof. Suppose that $(x_k, p_k) \in \mathcal{H}_k^\xi$. From (6.13) we know that

$$\sigma_r^2(T_k, S_k) = \epsilon^2 \Sigma_r^{(2)}[\wp_k, \bar{x}](X_k(S_k)) \quad (7.42)$$

where, by (5.27),

$$\lim_{\epsilon \rightarrow 0} \frac{\Sigma_r^{(2)}[\wp_k, \bar{x}](X_k(S_k))}{[P_k(S_k) - \wp_k^*(X_k(S_k))]^2} [\wp_k(\bar{x}) - \wp_k^*(\bar{x})]^2 (\bar{x} - 2(k-1)\pi)^{2(1+\theta)} \in (0, +\infty) \quad (7.43)$$

then, from (6.7) it follows that

$$\lim_{\epsilon \rightarrow 0} \frac{|P_k(S_k) - \wp_k^*(X_k(S_k))|}{\sigma_r(T_k, S_k) |\wp_k(\bar{x}) - \wp_k^*(\bar{x})|} \frac{\epsilon}{\eta_\epsilon^{(1+\theta)}} \in (0, +\infty) \quad (7.44)$$

from which the result. \square

Lemma 7.11. *There exist $C, c, c' > 0$ such that, for any $\zeta > 0$, ξ, ϵ small enough,*

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \{c\eta_\epsilon < 2k\pi - X_k(S_k) \leq c'\eta_\epsilon\} \geq 1 - e^{-C\epsilon^{-2\zeta}} \quad (7.45)$$

Proof. (7.45) follows from Lemma 7.6 and the monotonicity of $X_k(t)$. \square

Proof of Proposition 4.6. From Lemma 7.11 the hypothesis (5.3) holds for the couple $\bar{x}, X_k(S_k)$ with probability larger than $1 - e^{-C\epsilon^{-2\zeta}}$ then (4.13) directly follows from (7.42) and (5.44). \square

Proof of Proposition 4.7. (4.14) easily follows from (7.41) and (6.9) since, from (3.19) and (3.20) we have $\epsilon^{-1}\eta_\epsilon^{1+\theta}\bar{\sigma}_\epsilon = \epsilon^\theta$. (4.27) is a direct consequence of (4.14) and (4.13). \square

Let us define the processes

$$Z_k(t) = \wp_k^*(X_k(t)) - \lambda_-(X_k(t) - 2k\pi), \quad V_k(t) = \wp_k^*(X_k(t)) - \lambda_+(X_k(t) - 2(k-1)\pi)$$

we have the following result.

Lemma 7.12. *Let $(x_k, p_k) \in \mathcal{H}_k^\xi$ for some $\xi > 0$, then there exists $c > 0$ such that*

$$|V_k(T_k)| \leq c\eta_\epsilon^2 \quad \mathbf{P}_{T_k, x_k, p_k} \text{ a.s.} \quad (7.46)$$

There exist $C, c' > 0$ such that, for any $\xi > 0$, ζ, ϵ small enough,

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \{|Z_k(S_k)| \leq c'\eta_\epsilon^2\} \geq 1 - e^{-C\epsilon^{-2\zeta}} \quad (7.47)$$

Proof. Recalling that $X_k(T_k) = x(T_k) = \bar{x}$, (7.46) follows from (5.1) and the left hand side of (6.7), whereas (7.47) follows from (5.2) and (7.45). \square

Lemma 7.13. *There exist $C, c > 0$ such that, for any $\zeta > 0$, ξ, ϵ small enough,*

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ |\omega_k(S_k) - \lambda_-| \leq c \frac{\bar{\sigma}_\epsilon}{\eta_\epsilon} \right\} \geq 1 - e^{-C\epsilon^{-2\zeta}} \quad (7.48)$$

Proof. We recall that $\omega_k(S_k) = \frac{d}{dx} \wp_k(X_k(S_k))$. Suppose that $(x_k, p_k) \in \mathcal{H}_k^\xi$, ξ small enough then, by Lemma 7.11 the hypothesis (5.3) holds for the couple $\bar{x}, X_k(S_k)$ with probability larger than $1 - e^{-C\epsilon^{-2\zeta}}$. Hence we can apply (5.23) and (6.9) and obtain that there exist $c, C > 0$ such that for any ζ, ϵ small enough,

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ \left| \omega_k(S_k) - \frac{d}{dx} \wp_k^*(X_k(S_k)) \right| \leq c \frac{\bar{\sigma}_\epsilon}{\eta_\epsilon} \epsilon^{-\xi} \right\} \geq 1 - e^{-C\epsilon^{-2\zeta}} \quad (7.49)$$

on the other hand, by (5.2), there exist $c', C > 0$ such that, for any ζ, ϵ small enough,

$$\mathbf{1}_{(x_k, p_k) \in \mathcal{H}_k^\xi} \mathbf{P}_{T_k, x_k, p_k} \left\{ \left| \frac{d}{dx} \wp_k^*(X_k(S_k)) - \lambda_- \right| \leq c' \eta_\epsilon \right\} \geq 1 - e^{-C\epsilon^{-2\zeta}} \quad (7.50)$$

then (7.48) follows from (7.49) and (7.50) since $\eta_\epsilon^2 = o(\bar{\sigma}_\epsilon)$. \square

Proof of Proposition 4.10. Suppose that $(x_k, p_k) \in \mathcal{H}_k^\xi$ for some $\xi > 0$, then, from the definitions of $z_k(t)$ in (2.14), $\hat{z}_k(t)$ in (4.3) and $r_k(t)$ in (3.11) we have

$$z_k(S_k) - \hat{z}_k(S_k) = Z_k(S_k) + \lambda_- q_k(S_k) - \omega_k(S_k) y_k(S_k) \quad (7.51)$$

$$= Z_k(S_k) + r_k(S_k) + (\omega_k(S_k) - \lambda_-) y_k(S_k) \quad (7.52)$$

$$= \mathcal{R}_k(S_k) + Z_k(S_k) + (\omega_k(S_k) - \lambda_-) y_k(S_k) \quad (7.53)$$

where the last equivalence follows since $\bar{r}_k(S_k) = 0$. Hence (4.19) descends from (7.1), (7.9), (7.47) and Lemma 7.13.

Suppose now that $(\bar{x}_{k-1}, \bar{p}_{k-1}) \in \mathcal{K}_{k-1}^\xi$ for some $\xi > 0$. From the definitions of $v_k(t)$ in (2.14) and $\mathcal{V}_k(t)$ in (3.4) we have

$$\begin{aligned} p(T_k) - \wp_k^*(x(T_k)) - \bar{v}_{k-1}(T_k) &= \mathcal{V}_{k-1}(T_k) - V_k(T_k) + \lambda_+ y_k(T_k) + \wp_k^*(X_k(T_k)) - \wp_k^*(x(T_k)) \\ &= \mathcal{V}_{k-1}(T_k) - V_k(T_k) \end{aligned}$$

where the last equality follows since $X_k(T_k) = x(T_k) = x_k$. Then (4.20) descends from (4.17) and (7.46). \square

8 Appendix

In the present Appendix we provide a Gaussian Inequality and a comparison result.

Marcus-Shepp inequality for Gaussian processes. There is a classical result of Landau and Shepp [4] and Marcus and Shepp [5] that gives an estimate for the supremum of a general centered Gaussian process. If $X(t)$ is an a.s. bounded, centered Gaussian process of variance $\sigma^2(t)$, then,

$$\lim_{\lambda \rightarrow \infty} \frac{1}{\lambda^2} \ln \mathbf{P} \left\{ \sup_{t \in I} X(t) \geq \lambda \right\} = -\frac{1}{2\sigma_I^2} \quad \text{with} \quad \sigma_I^2 := \sup_{t \in I} \sigma^2(t) \quad (8.1)$$

An immediate consequence of (8.1) is that for any λ large enough, δ small enough,

$$\mathbf{P} \left\{ \sup_{t \in I} X(t) \geq \lambda \right\} \leq e^{-\frac{\lambda^2}{2\sigma_I^2} (1-\delta)} \quad (8.2)$$

moreover, by symmetry we have

$$\mathbf{P} \left\{ \sup_{t \in I} |X(t)| \geq \lambda \right\} \leq 2\mathbf{P} \left\{ \sup_{t \in I} X(t) \geq \lambda \right\}. \quad (8.3)$$

By applying the result to the process $X(t)/\sigma(t)$ we get

$$\mathbf{P} \left\{ \sup_t \frac{|X(t)|}{\sigma(t)} \geq \lambda \right\} \leq 2e^{-\frac{\lambda^2}{2} (1-\delta)} \quad (8.4)$$

for λ large enough, δ small enough.

Comparison with Gaussian Processes. In our proofs we repeatedly make use of a comparison argument comparing the solution of a linear SDE with the solution of a more general SDE, let us see. Let X_t be a solution of the problem

$$dX_t = (a(t)X_t + b(t))dt + \xi dw_t, \quad (8.5)$$

with $a, b : \mathbb{R}^+ \rightarrow \mathbb{R}$ bounded on bounded intervals and $\xi \in \mathbb{R}$, then $X(t)$ is a Gaussian process of the form

$$X(t) = X(t_0) e^{\int_{t_0}^t a(s) ds} + \int_{t_0}^t b(s) e^{\int_s^t a(s') ds'} ds + \xi \int_{t_0}^t e^{\int_s^t a(s') ds'} dw_s.$$

Consider, now, the processes $x(t)$ solution of

$$dx_t = c(x_t, t)dt + \xi dw_t$$

with the same noise of (8.5), $c : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ globally Lipschitz.

Lemma 8.1. *For $X(t), x(t)$ as above we define $\delta_t := c(X_t, t) - [a(t)X_t + b(t)]$, $\Delta_t := X_t - x_t$, and let $\tau \in \mathbb{R}^+$ be a generic random variable. Suppose*

$$\text{sign}(\Delta_\tau) = \text{sign}(\delta_\tau) \quad \text{or} \quad \Delta_\tau = 0,$$

then

$$\text{sign}(\Delta_t) = \text{sign}(\delta_t) \quad \text{for any } \tau \leq t \leq \inf\{s \geq \tau : \delta_s = 0\} \quad a.s.$$

Proof. We have

$$d\Delta_t = (a(t)\Delta_t + \delta_t)dt$$

thus, for any $\tau \geq 0$

$$\Delta(t) = \Delta(\tau) e^{\int_\tau^t a(s) ds} + \int_\tau^t \delta(s) e^{\int_s^t a(s') ds'} ds$$

then follows the result. □

References

- [1] Freidlin M. I., Functional Integration and Partial Differential Equations *Annals of Mathematics Studies, Princeton, 1985*
- [2] Freidlin M. I., Wentzel A.D., Random Perturbations of Dynamical Systems *Springer-Verlag 1984*
- [3] Li W. V., Small Deviations for Gaussian Markov Processes Under the Sup-Norm, *Journal of Theoretical Probability*, 4 **12**, 1999, 971-984
- [4] Landau H., Shepp L. A., On the Supremum of a Gaussian Process, *Sankhya A*, **32**, 1970, 369-378
- [5] Marcus M. B., Shepp L. A. Sample behaviour of Gaussian Processes, *Proceedings of the 6th Berkeley Symposium on Mathematics, Statistic and Probability*, Vol.2, University of California Press, Berkeley, CA, 1972, 423-442.

- [6] Perez-Mato J.M., Blaha P., Schwarz K., Aroyo M., Orobengoa D., Etxebarria I., Garcia A., Multiple instabilities in $\text{Bi}_4\text{Ti}_3\text{O}_{12}$: A ferroelectric beyond the soft-mode paradigm *Phys.Rev. B* **77**, 2008, 184104
- [7] Perez-Mato J. M., Ribeiro J. L., Petricek V., Aroyo M. I., Magnetic superspace groups and symmetry constraints in incommensurate magnetic phases, *J. Phys.: Condens. Matter* **24**, 2012, 163201
- [8] Risken H. The Fokker-Planck Equation. Methods of Solution and Applications *New York: Springer 1996*
- [9] Risken H., Vollmer H.D. Brownian Motion in Periodic Potentials in the Low-Friction-Limit; Nonlinear Response to an External Force *Z. Physik B* **35**, 1979, 177-184.